Summary of Session : 1

$$\mathbf{A}\cdot\mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3$$

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\alpha = AB\cos\alpha$$

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\alpha\,\mathbf{a}_N = AB\sin\alpha\,\mathbf{a}_N$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\mathbf{a}_N = \frac{\mathbf{A}\times\mathbf{B}}{AB\sin\alpha} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B}$$

$$\begin{aligned}\mathbf{D}\cdot\mathbf{A}\times(\mathbf{B}+\mathbf{C}) &= (\mathbf{B}+\mathbf{C})\cdot(\mathbf{D}\times\mathbf{A}) = \mathbf{B}\cdot(\mathbf{D}\times\mathbf{A}) + \mathbf{C}\cdot(\mathbf{D}\times\mathbf{A}) \\ &= \mathbf{D}\cdot\mathbf{A}\times\mathbf{B} + \mathbf{D}\cdot\mathbf{A}\times\mathbf{C} = \mathbf{D}\cdot(\mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C})\end{aligned}$$

Given three vectors

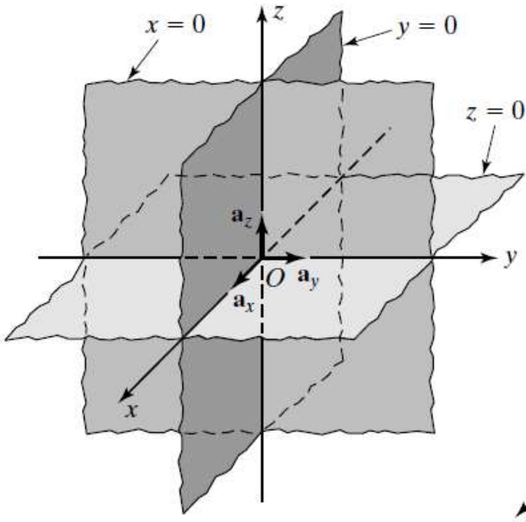
$$\begin{aligned}\mathbf{A} &= 3\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3 \\ \mathbf{B} &= \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \\ \mathbf{C} &= \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3\end{aligned}$$

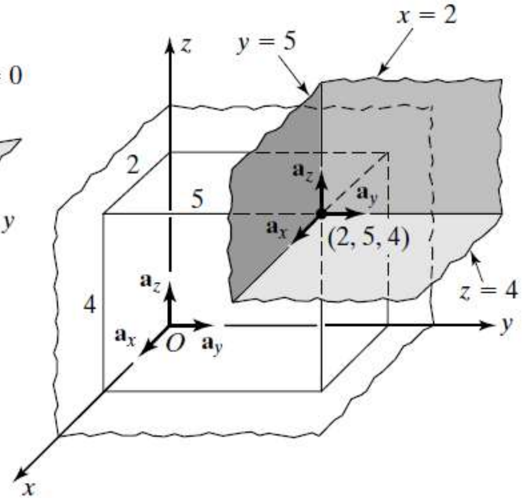
Find the following: **(a)** $|\mathbf{A} + \mathbf{B} - 4\mathbf{C}|$; **(b)** unit vector along $(\mathbf{A} + 2\mathbf{B} - \mathbf{C})$; **(c)** $\mathbf{A}\cdot\mathbf{C}$; **(d)** $\mathbf{B}\times\mathbf{C}$; and **(e)** $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}$.

Ans. **(a)** 13; **(b)** $(2\mathbf{a}_1 + \mathbf{a}_2 - 2\mathbf{a}_3)/3$; **(c)** 10; **(d)** $5\mathbf{a}_1 - 4\mathbf{a}_2 + \mathbf{a}_3$; **(e)** 8.

CARTESIAN COORDINATE SYSTEM

$$x = \text{constant}$$
$$y = \text{constant}$$
$$z = \text{constant}$$





$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{(x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}}$$

Differential length vector.

$$d\mathbf{l} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$$

$$\mathbf{a}_n = \frac{d\mathbf{l}_1 \times d\mathbf{l}_2}{|d\mathbf{l}_1 \times d\mathbf{l}_2|}$$

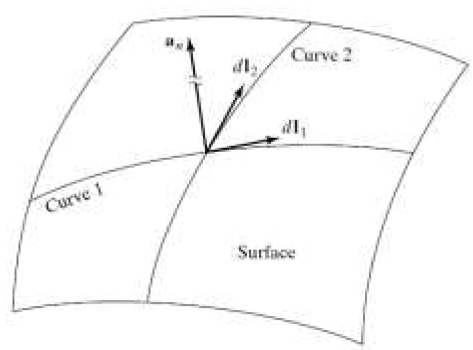
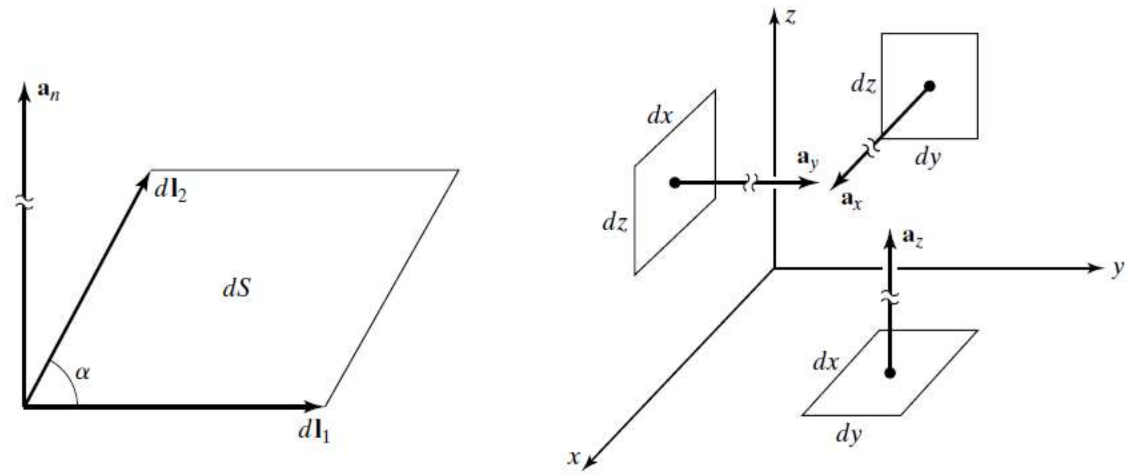


FIGURE 1.10

Finding the unit vector normal to a surface by using differential length vectors.

Differential surface vector. $dS = dl_1\, dl_2 \sin\alpha = |d\mathbf{l}_1 \times d\mathbf{l}_2|$

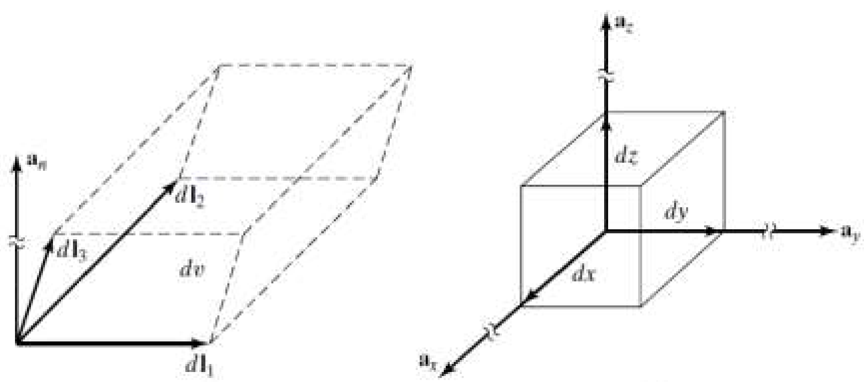
$$d\mathbf{S} = \pm d\mathbf{l}_1 \times d\mathbf{l}_2$$



(a) (b)

Differential volume.

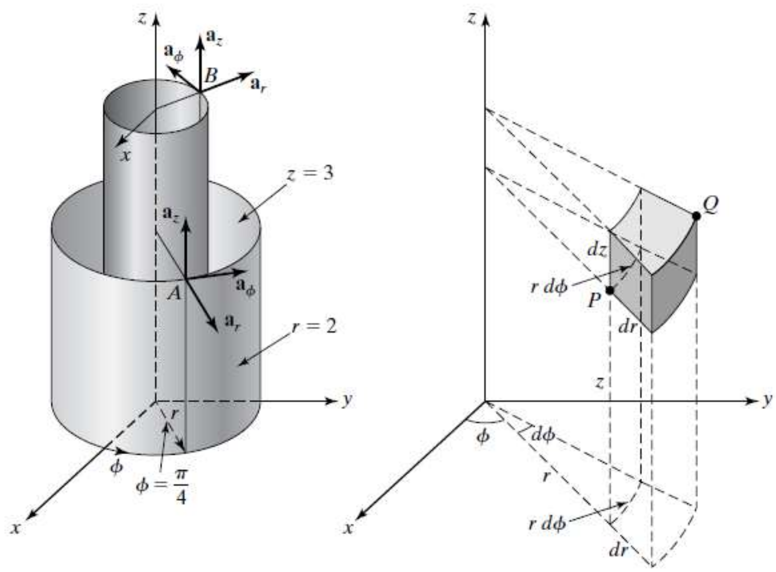
$$\begin{aligned} dv &= \text{area of the base of the parallelepiped} \times \text{height of the parallelepiped} \\ &= |d\mathbf{l}_1 \times d\mathbf{l}_2|\,|d\mathbf{l}_3 \cdot \mathbf{a}_n| \\ &= |d\mathbf{l}_1 \times d\mathbf{l}_2| \frac{|d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2|}{|d\mathbf{l}_1 \times d\mathbf{l}_2|} \\ &= |d\mathbf{l}_3 \cdot d\mathbf{l}_1 \times d\mathbf{l}_2| \end{aligned}$$



(a) (b)

CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS

$$
\begin{aligned}
r &= \text{constant} \\
\phi &= \text{constant} \\
z &= \text{constant}
\end{aligned}
$$



(a) (b)

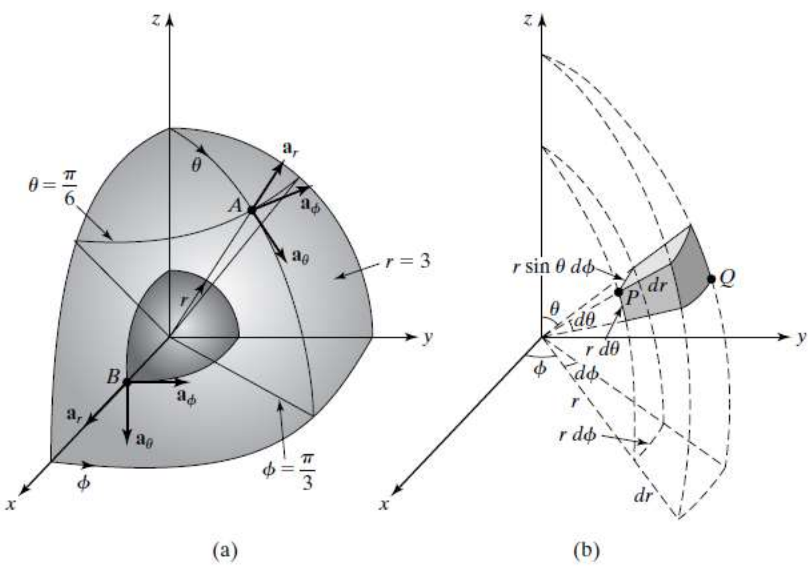
$$
d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z
$$

$$
\begin{aligned}
\pm r\,d\phi\,\mathbf{a}_\phi \times dz\,\mathbf{a} &= \pm r\,d\phi\,dz\,\mathbf{a}_r \\
\pm dz\,\mathbf{a}_z \times dr\,\mathbf{a}_r &= \pm dr\,dz\,\mathbf{a}_\phi \\
\pm dr\,\mathbf{a}_r \times r\,d\phi\,\mathbf{a}_\phi &= \pm r\,dr\,d\phi\,\mathbf{a}_z
\end{aligned}
$$

$$
dv = (dr)(r\,d\phi)(dz) = r\,dr\,d\phi\,dz
$$

Spherical coordinate system

$$\begin{aligned} r &= \text{constant} \\ \theta &= \text{constant} \\ \phi &= \text{constant} \end{aligned}$$



$$d\mathbf{l} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

$$\begin{aligned} \pm r\,d\theta\,\mathbf{a}_\theta \times r\sin\theta\,d\phi\,\mathbf{a}_\phi &= \pm r^2\sin\theta\,d\theta\,d\phi\,\mathbf{a}_r \\ \pm r\sin\theta\,d\phi\,\mathbf{a}_\phi \times dr\,\mathbf{a}_r &= \pm r\sin\theta\,dr\,d\phi\,\mathbf{a}_\theta \\ \pm dr\,\mathbf{a}_r \times r\,d\theta\,\mathbf{a}_\theta &= \pm r\,dr\,d\theta\,\mathbf{a}_\phi \end{aligned}$$

$$dv = (dr)(r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Conversions between coordinate systems

$$x = r_c \cos\phi \qquad y = r_c \sin\phi \qquad z = z$$
$$x = r_s \sin\theta \cos\phi \qquad y = r_s \sin\theta \sin\phi \qquad z = r_s \cos\theta$$

$$r_c = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\frac{y}{x} \qquad z = z$$
$$r_s = \sqrt{x^2 + y^2 + z^2} \qquad \theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$\mathbf{a}_{rc} \cdot \mathbf{a}_x = \cos\phi \qquad \mathbf{a}_{rc} \cdot \mathbf{a}_y = \sin\phi \qquad \mathbf{a}_{rc} \cdot \mathbf{a}_z = 0$$
$$\mathbf{a}_{\phi} \cdot \mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi} \cdot \mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi} \cdot \mathbf{a}_z = 0$$
$$\mathbf{a}_z \cdot \mathbf{a}_x = 0 \qquad \mathbf{a}_z \cdot \mathbf{a}_y = 0 \qquad \mathbf{a}_z \cdot \mathbf{a}_z = 1$$

$$\mathbf{a}_{rs} \cdot \mathbf{a}_x = \sin\theta \cos\phi \qquad \mathbf{a}_{rs} \cdot \mathbf{a}_y = \sin\theta \sin\phi \qquad \mathbf{a}_{rs} \cdot \mathbf{a}_z = \cos\theta$$
$$\mathbf{a}_{\theta} \cdot \mathbf{a}_x = \cos\theta \cos\phi \qquad \mathbf{a}_{\theta} \cdot \mathbf{a}_y = \cos\theta \sin\phi \qquad \mathbf{a}_{\theta} \cdot \mathbf{a}_z = -\sin\theta$$
$$\mathbf{a}_{\phi} \cdot \mathbf{a}_x = -\sin\phi \qquad \mathbf{a}_{\phi} \cdot \mathbf{a}_y = \cos\phi \qquad \mathbf{a}_{\phi} \cdot \mathbf{a}_z = 0$$

Example 1.4 Conversion of a vector from Cartesian to spherical coordinates

Let us consider the vector $3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ at the point (3, 4, 5) and convert it to one in spherical coordinates.

First, from the relationships (1.47b), we obtain the spherical coordinates of the point (3, 4, 5) to be

$$r_s = \sqrt{3^2 + 4^2 + 5^2} = 5\sqrt{2}$$

$$\theta = \tan^{-1}\frac{\sqrt{3^2 + 4^2}}{5} = \tan^{-1} 1 = 45^\circ$$

$$\phi = \tan^{-1}\tfrac{4}{3} = 53.13^\circ$$

Then, noting from the relationships (1.49) that

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_x \cdot \mathbf{a}_{rs}) & (\mathbf{a}_x \cdot \mathbf{a}_\theta) & (\mathbf{a}_x \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_y \cdot \mathbf{a}_{rs}) & (\mathbf{a}_y \cdot \mathbf{a}_\theta) & (\mathbf{a}_y \cdot \mathbf{a}_\phi) \\ (\mathbf{a}_z \cdot \mathbf{a}_{rs}) & (\mathbf{a}_z \cdot \mathbf{a}_\theta) & (\mathbf{a}_z \cdot \mathbf{a}_\phi) \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

we obtain at the point under consideration

$$\begin{bmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{bmatrix} = \begin{bmatrix} 0.3\sqrt{2} & 0.3\sqrt{2} & -0.8 \\ 0.4\sqrt{2} & 0.4\sqrt{2} & 0.6 \\ 0.5\sqrt{2} & -0.5\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{a}_{rs} \\ \mathbf{a}_\theta \\ \mathbf{a}_\phi \end{bmatrix}$$

$$\begin{aligned} 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z &= 3(0.3\sqrt{2}\mathbf{a}_{rs} + 0.3\sqrt{2}\mathbf{a}_\theta - 0.8\mathbf{a}_\phi) \\ &\quad + 4(0.4\sqrt{2}\mathbf{a}_{rs} + 0.4\sqrt{2}\mathbf{a}_\theta + 0.6\mathbf{a}_\phi) \\ &\quad + 5(0.5\sqrt{2}\mathbf{a}_{rs} - 0.5\sqrt{2}\mathbf{a}_\theta) \\ &= 5\sqrt{2}\mathbf{a}_{rs} \end{aligned}$$

Train more by solving below Questions :

D1.7. Convert into Cartesian coordinates each of the following points: **(a)** $(2, 5\pi/6, 3)$ in cylindrical coordinates; **(b)** $(4, 4\pi/3, -1)$ in cylindrical coordinates; **(c)** $(4, 2\pi/3, \pi/6)$ in spherical coordinates; and **(d)** $(\sqrt{8}, \pi/4, \pi/3)$ in spherical coordinates.
Ans. **(a)** $(-\sqrt{3}, 1, 3)$; **(b)** $(-2, -2\sqrt{3}, -1)$; **(c)** $(3, \sqrt{3}, -2)$; **(d)** $(1, \sqrt{3}, 2)$.

D1.8. Convert into cylindrical coordinates the following points specified in Cartesian coordinates: **(a)** $(-2, 0, 1)$; **(b)** $(1, -\sqrt{3}, -1)$; and **(c)** $(-\sqrt{2}, -\sqrt{2}, 3)$.
Ans. **(a)** $(2, \pi, 1)$; **(b)** $(2, 5\pi/3, -1)$; **(c)** $(2, 5\pi/4, 3)$.

D1.9. Convert into spherical coordinates the following points specified in Cartesian coordinates: **(a)** $(0, -2, 0)$; **(b)** $(-3, \sqrt{3}, 2)$; and **(c)** $(-\sqrt{2}, 0, -\sqrt{2})$.
Ans. **(a)** $(2, \pi/2, 3\pi/2)$; **(b)** $(4, \pi/3, 5\pi/6)$; **(c)** $(2, 3\pi/4, \pi)$.