Session 10

CURL AND DIVERGENCE

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

A. Curl

$$\nabla \times \mathbf{H} = \mathbf{J} \qquad (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S} = \mathbf{J} \cdot \Delta \mathbf{S} \qquad \oint_C \mathbf{H} \cdot d\mathbf{l} = \mathbf{J} \cdot \Delta \mathbf{S}$$

$$(\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l} \qquad (\nabla \times \mathbf{H}) \cdot \Delta S\, \mathbf{a}_n = \oint_C \mathbf{H} \cdot d\mathbf{l}$$

$$(\nabla \times \mathbf{H}) \cdot \mathbf{a}_n = \frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S} \qquad |\nabla \times \mathbf{H}| = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max}$$

$$\nabla \times \mathbf{H} = \left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n \qquad \nabla \times \mathbf{H} = \lim_{\Delta S \to 0}\left[\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n$$

$$\nabla \times \mathbf{A} = \lim_{\Delta S \to 0}\left[\frac{\oint_C \mathbf{A} \cdot d\mathbf{l}}{\Delta S}\right]_{\max} \mathbf{a}_n$$

Equation (3.50) tells us that to find the curl of a vector at a point in that vector field, we first consider an infinitesimal surface at that point and compute the closed line integral or circulation of the vector around the periphery of this surface by orienting the surface such that the circulation is maximum. We then

Physical interpretation of curl

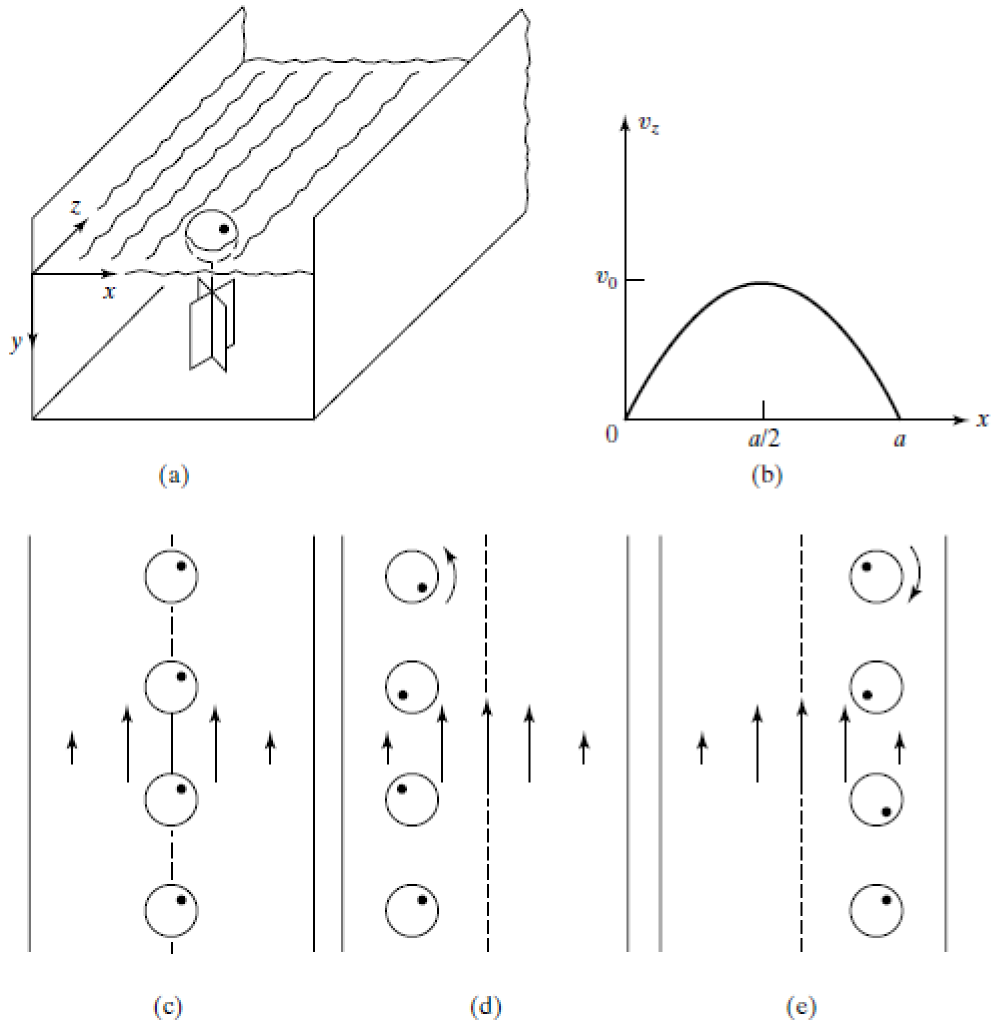

FIGURE 3.8

For explaining the physical interpretation of curl using the curl meter.

$$\mathbf{v} = v_z(x)\mathbf{a}_z = v_0 \sin\frac{\pi x}{a}\mathbf{a}_z$$

its curl is given by

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v_z \end{vmatrix}$$

$$= -\frac{\partial v_z}{\partial x}\mathbf{a}_y$$

$$= -\frac{\pi v_0}{a}\cos\frac{\pi x}{a}\mathbf{a}_y$$

B. Divergence

$$\nabla \cdot \mathbf{D} = \rho \qquad (\nabla \cdot \mathbf{D})\,\Delta v = \rho\,\Delta v$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \rho\,\Delta v \qquad (\nabla \cdot \mathbf{D})\,\Delta v = \oint_S \mathbf{D} \cdot d\mathbf{S} \qquad \nabla \cdot \mathbf{D} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\nabla \cdot \mathbf{D} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{D} \cdot d\mathbf{S}}{\Delta v}$$

$$\boxed{\nabla \cdot \mathbf{A} = \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta v}} \qquad (3.57)$$

Equation (3.57) tells us that to find the divergence of a vector at a point in that vector field, we first consider an infinitesimal volume at that point and compute the surface integral of the vector over the surface bounding that volume, that is, the outward flux of the vector field from that volume. We then di-

Physical interpretation of divergence

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

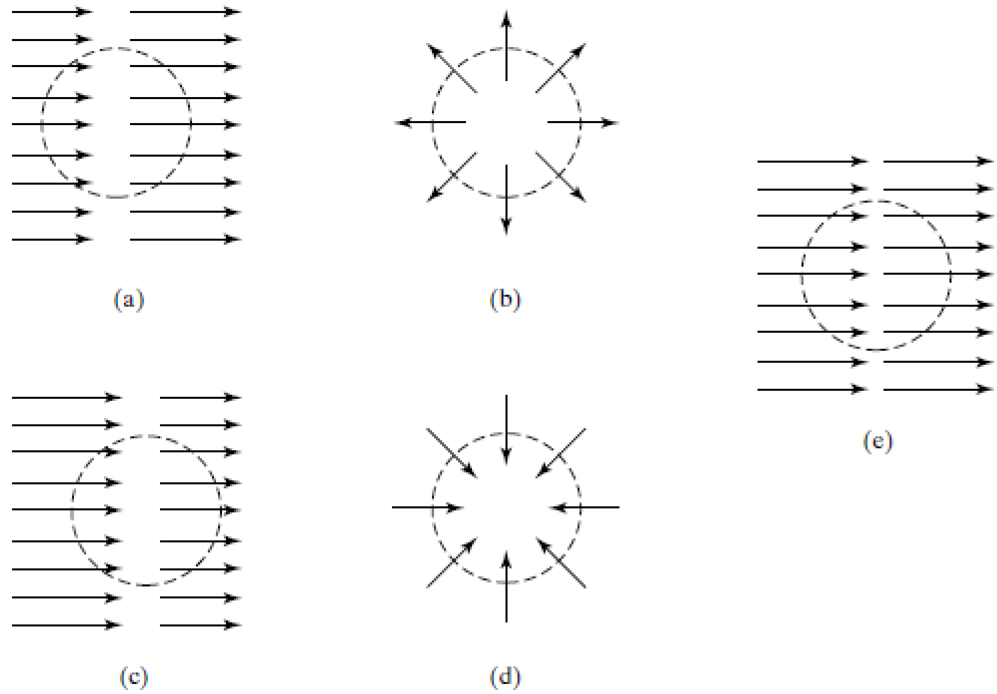

FIGURE 3.9

For explaining the physical interpretation of divergence using the divergence meter.

C. Stokes' and Divergence Theorems

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \oint_C \mathbf{H} \cdot d\mathbf{l}$$

Divergence theorem

$$\int_V (\nabla \cdot \mathbf{D})\, dv = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$\boxed{\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{A})\, dv}$$

D3.7. With the aid of the curl meter, determine if the z-component of the curl of the vector field $\mathbf{A} = (x^2 - 4)\mathbf{a}_y$ is positive, zero, or negative at each of the following points: **(a)** $(2, -3, 1)$; **(b)** $(0, 2, 4)$; and **(c)** $(-1, 2, -1)$.

Ans. **(a)** positive; **(b)** zero; **(c)** negative.

D3.8. With the aid of the divergence meter, determine if the divergence of the vector field $\mathbf{A} = (x - 2)^2\mathbf{a}_x$ is positive, zero, or negative at each of the following points: **(a)** $(2, 4, 3)$; **(b)** $(1, 1, -1)$; and **(c)** $(3, -1, 4)$.

Ans. **(a)** zero; **(b)** negative; **(c)** positive.

D3.9. Using Stokes' theorem, find the absolute value of the line integral of the vector field $(x\mathbf{a}_y + \sqrt{3}y\mathbf{a}_z)$ around each of the following closed paths: **(a)** the perimeter of a square of sides 2 m lying in the xy-plane; **(b)** a circular path of radius $1/\sqrt{\pi}$ m lying in the xy-plane; and **(c)** the perimeter of an equilateral triangle of sides 2 m lying in the yz-plane.

Ans. **(a)** 4; **(b)** 1; **(c)** 3.