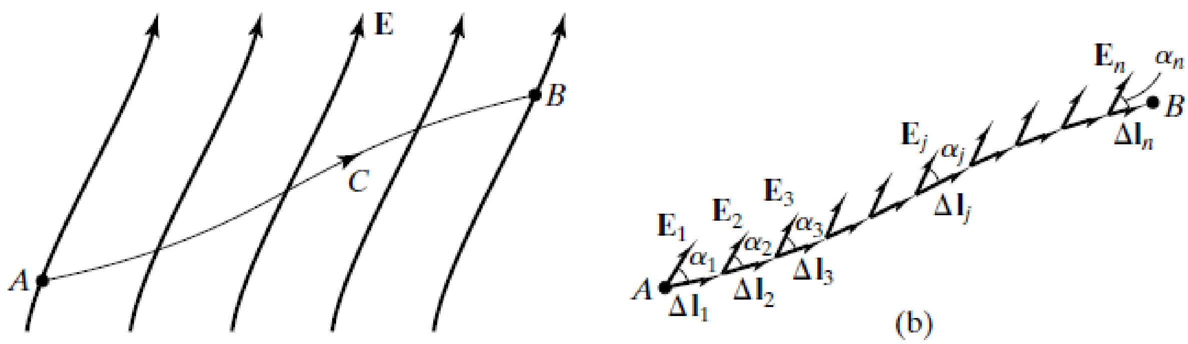
Session 5 :

CHAPTER 2

Maxwell's Equations in Integral Form

THE LINE INTEGRAL



(a) (b)

$$\Delta W_j = qE_j \cos \alpha_j \, \Delta l_j$$

$$\begin{aligned} W_{AB} &= \Delta W_1 + \Delta W_2 + \Delta W_3 + \cdots + \Delta W_n \\ &= qE_1 \cos \alpha_1 \, \Delta l_1 + qE_2 \cos \alpha_2 \, \Delta l_2 + qE_3 \cos \alpha_3 \, \Delta l_3 + \cdots \\ &\quad + qE_n \cos \alpha_n \, \Delta l_n \\ &= q \sum_{j=1}^{n} E_j \cos \alpha_j \, \Delta l_j \\ &= q \sum_{j=1}^{n} (E_j)(\Delta l_j) \cos \alpha_j \end{aligned}$$

$$\boxed{W_{AB} = q \sum_{j=1}^{n} \mathbf{E}_j \cdot \Delta \mathbf{l}_j}$$

For a numerical example, let us consider the electric field given by

$$\mathbf{E} = y\mathbf{a}_y$$

and determine the work done by the field in the movement of 3 μC of charge from the point $A(0, 0, 0)$ to the point $B(1, 1, 0)$ along the parabolic path $y = x^2$, $z = 0$ shown in Fig. 2.2(a).

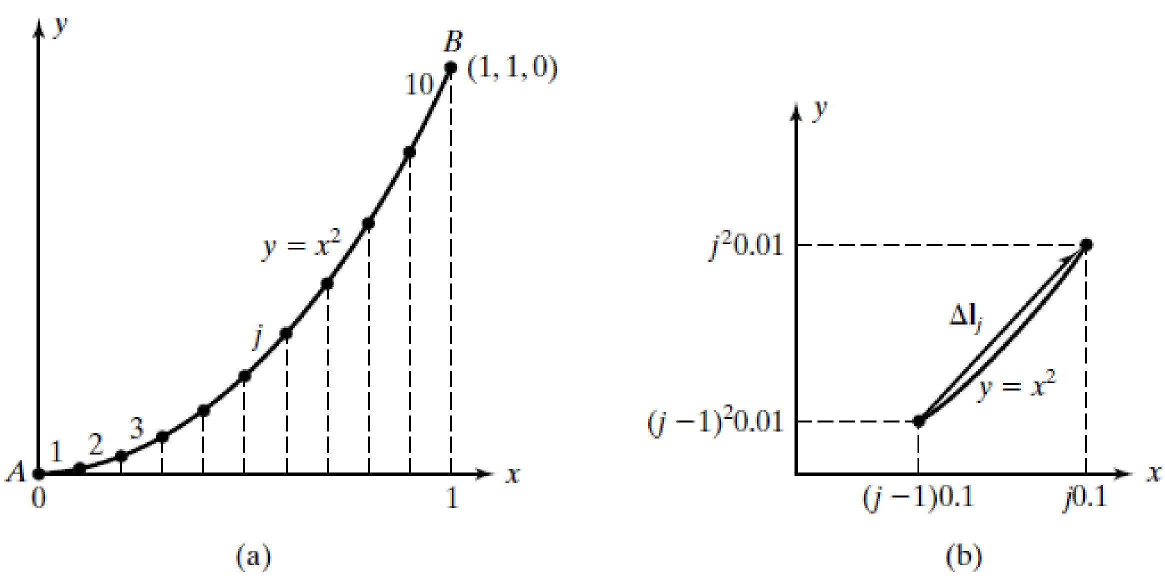


FIGURE 2.2

(a) Division of the path $y = x^2$ from $A(0, 0, 0)$ to $B(1, 1, 0)$ into 10 segments. (b) Length vector corresponding to the jth segment of part (a) approximated as a straight line.

$$\mathbf{E}_j = (j - 1)^2 0.01\mathbf{a}_y$$

$$\Delta\mathbf{l}_j = 0.1\mathbf{a}_x + [j^2 - (j - 1)^2]0.01\mathbf{a}_y = 0.1\mathbf{a}_x + (2j - 1)0.01\mathbf{a}_y$$

$$W_{AB} = 3 \times 10^{-6} \sum_{j=1}^{10} \mathbf{E}_j \cdot \Delta\mathbf{l}_j$$

$$= 3 \times 10^{-6} \sum_{j=1}^{10} [(j - 1)^2 0.01\mathbf{a}_y] \cdot [0.1\mathbf{a}_x + (2j - 1)0.01\mathbf{a}_y]$$

$$= 3 \times 10^{-10} \sum_{j=1}^{10} (j - 1)^2(2j - 1)$$

$$= 3 \times 10^{-10}[0 + 3 + 20 + 63 + 144 + 275 + 468 + 735 + 1088 + 1539]$$

$$= 3 \times 10^{-10} \times 4335 \text{ J} = 1.3005\ \mu\text{J}$$

$$\boxed{W_{AB} = q \int_A^B \mathbf{E} \cdot d\mathbf{l}}$$

$$dy = 2x\,dx \qquad dz = 0 \qquad \begin{aligned} d\mathbf{l} &= dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \\ &= dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y \end{aligned}$$

$$\begin{aligned} \mathbf{E} \cdot d\mathbf{l} &= y\mathbf{a}_y \cdot (dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y) \\ &= x^2\mathbf{a}_y \cdot (dx\,\mathbf{a}_x + 2x\,dx\,\mathbf{a}_y) \\ &= 2x^3\,dx \end{aligned}$$

$$\begin{aligned} W_{AB} &= q \int_{(0,0,0)}^{(1,1,0)} \mathbf{E} \cdot d\mathbf{l} = 3 \times 10^{-6} \int_0^1 2x^3\,dx \\ &= 3 \times 10^{-6} \left[\frac{2x^4}{4}\right]_0^1 = 1.5\ \mu\text{J} \end{aligned}$$

Voltage defined

$$\boxed{V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{l}}$$

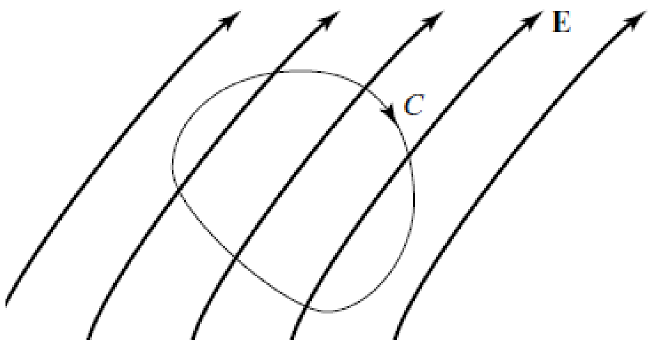


FIGURE 2.3
Closed path C in a region of electric field.

$\oint_C \mathbf{E} \cdot d\mathbf{l}$: Electromotive Force (emf)

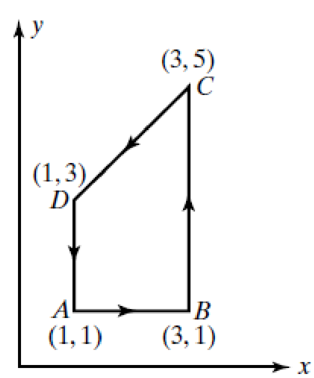
Evaluation of line integral around a closed path

$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} = \int_A^B \mathbf{F} \cdot d\mathbf{l} + \int_B^C \mathbf{F} \cdot d\mathbf{l} + \int_C^D \mathbf{F} \cdot d\mathbf{l} + \int_D^A \mathbf{F} \cdot d\mathbf{l}$$

$$y = 1, \qquad dy = 0, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + (0)\mathbf{a}_y = dx\,\mathbf{a}_x$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x) = 0$$

$$\int_A^B \mathbf{F} \cdot d\mathbf{l} = 0$$



For the side BC,

$$x = 3, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y = dy\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (3\mathbf{a}_y) \cdot (dy\,\mathbf{a}_y) = 3\,dy$$

$$\int_B^C \mathbf{F} \cdot d\mathbf{l} = \int_1^5 3\,dy = 12$$

For the side CD,

$$y = 2 + x, \qquad dy = dx, \qquad d\mathbf{l} = dx\,\mathbf{a}_x + dx\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (x\mathbf{a}_y) \cdot (dx\,\mathbf{a}_x + dx\,\mathbf{a}_y) = x\,dx$$

$$\int_C^D \mathbf{F} \cdot d\mathbf{l} = \int_3^1 x\,dx = -4$$

For the side DA,

$$x = 1, \qquad dx = 0, \qquad d\mathbf{l} = (0)\mathbf{a}_x + dy\,\mathbf{a}_y$$

$$\mathbf{F} \cdot d\mathbf{l} = (\mathbf{a}_y) \cdot (dy\,\mathbf{a}_y) = dy$$

$$\int_D^A \mathbf{F} \cdot d\mathbf{l} = \int_3^1 dy = -2$$

Finally,

$$\oint_{ABCDA} \mathbf{F} \cdot d\mathbf{l} = 0 + 12 - 4 - 2 = 6$$

In this example, we found that the line integral of **F** around the closed path *C* is nonzero. The field is then said to be a *nonconservative field*. For a nonconservative field, the line integral between two points, say, *A* and *B*, is dependent on the path followed from *A* to *B*. To show this, let us consider the two paths *ACB* and *ADB*, as shown in Fig. 2.5. Then we can write

$$\begin{aligned}\oint_{ACBDA} \mathbf{F}\cdot d\mathbf{l} &= \int_{ACB} \mathbf{F}\cdot d\mathbf{l} + \int_{BDA} \mathbf{F}\cdot d\mathbf{l} \\ &= \int_{ACB} \mathbf{F}\cdot d\mathbf{l} - \int_{ADB} \mathbf{F}\cdot d\mathbf{l}\end{aligned} \tag{2.6}$$

It can be easily seen that if $\oint_{ACBDA}\mathbf{F}\cdot d\mathbf{l}$ is not equal to zero, then $\int_{ACB}\mathbf{F}\cdot d\mathbf{l}$ is not equal to $\int_{ADB}\mathbf{F}\cdot d\mathbf{l}$. The two integrals are equal only if $\oint_{ACBDA}\mathbf{F}\cdot d\mathbf{l}$ is equal to zero, which is the case for *conservative fields*. Examples of conservative fields are Earth's gravitational field and the static electric field. An example of nonconservative fields is the time-varying electric field. Thus, in a time-varying electric field, the voltage between two points *A* and *B* is dependent on the path followed to evaluate the line integral of **E** from *A* to *B*, whereas in a static electric field, the voltage, more commonly known as the *potential difference*, between two points *A* and *B* is uniquely defined because the line integral of **E** from *A* to *B* is independent of the path followed from *A* to *B*.

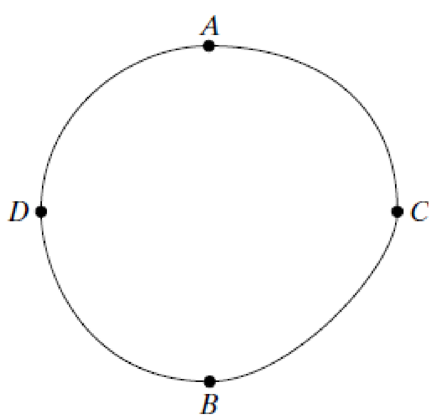


FIGURE 2.5

Two different paths from point *A* to point *B*.

D2.1. For each of the curves **(a)** $y = x^2, z = 0$, **(b)** $x^2 + y^2 = 2, z = 0$, and **(c)** $y = \sin 0.5\pi x, z = 0$ in a region of electric field $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y$, find the approximate value of the work done by the field in carrying a charge of 1 μC from the point $(1, 1, 0)$ to the neighboring point on the curve, whose x coordinate is 1.1, by evaluating $\mathbf{E} \cdot \Delta\mathbf{l}$ along a straight line path.

Ans. **(a)** 0.31 μJ; **(b)** -0.0112 μJ; **(c)** 0.0877 μJ.

D2.2. For $\mathbf{F} = y(\mathbf{a}_x + \mathbf{a}_y)$, find $\int \mathbf{F} \cdot d\mathbf{l}$ for the straight-line paths between the following pairs of points from the first point to the second point: **(a)** $(0, 0, 0)$ to $(2, 0, 0)$; **(b)** $(0, 2, 0)$ to $(2, 2, 0)$; and **(c)** $(2, 0, 0)$ to $(2, 2, 0)$.

Ans. **(a)** 0; **(b)** 4; **(c)** 2.

THE SURFACE INTEGRAL

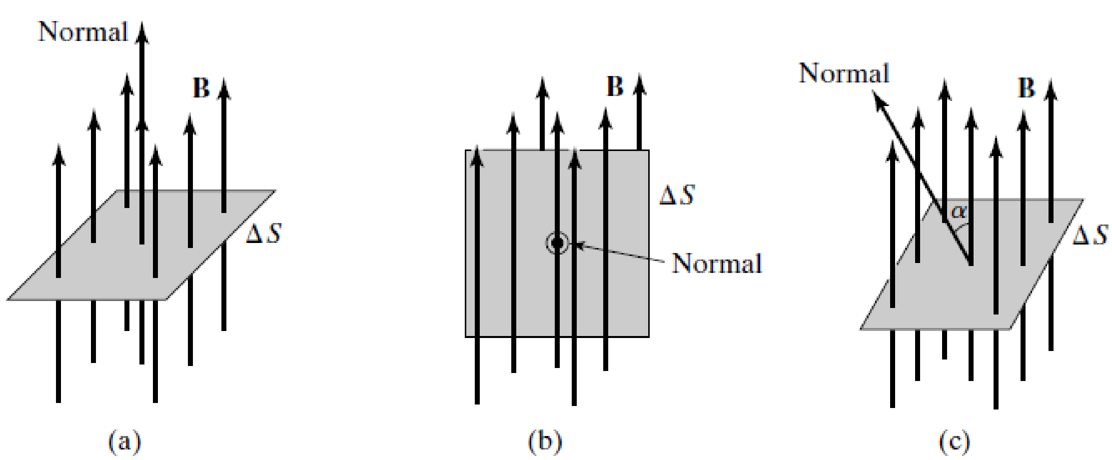
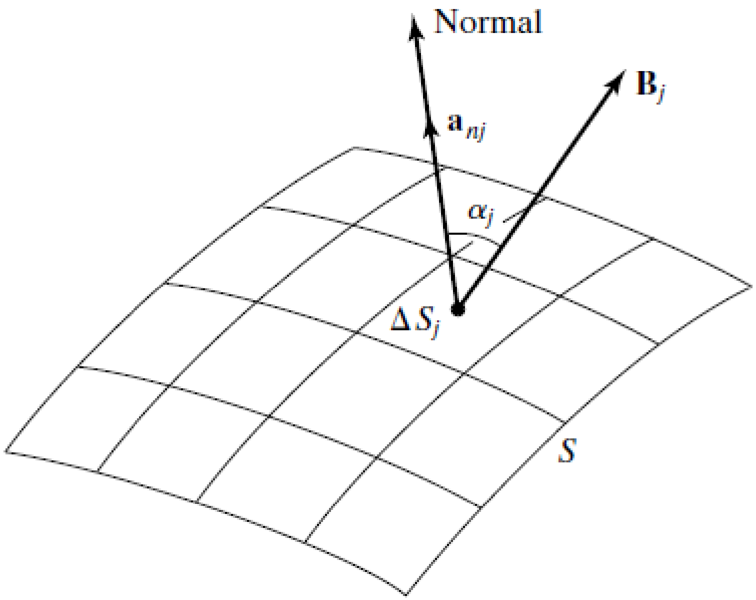


FIGURE 2.6

Infinitesimal surface ΔS in a magnetic field **B** oriented (a) normal to the field. (b) parallel to the field, and (c) with its normal making an angle a to the field.

$$\Delta\psi_j = B_j \cos\alpha_j \, \Delta S_j$$

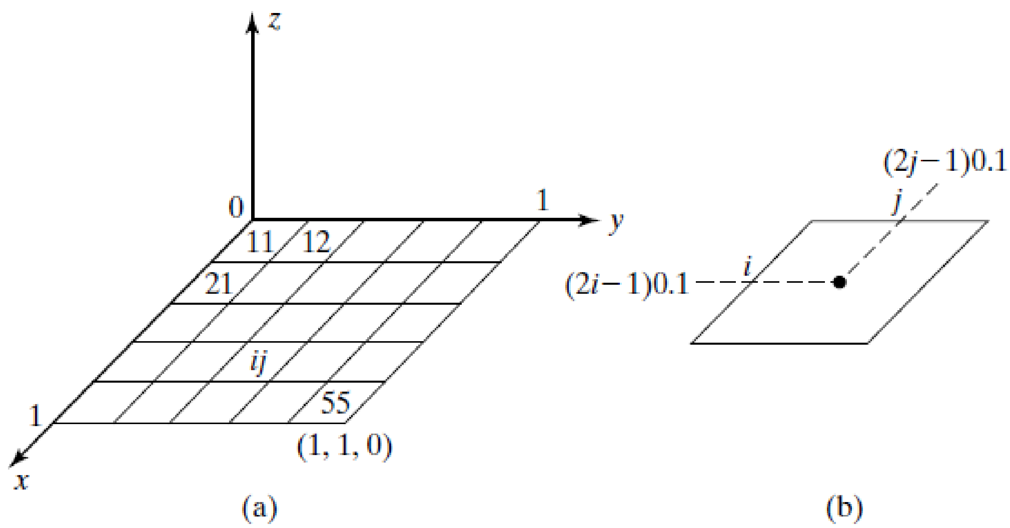


$$\begin{aligned}[\psi]_S &= \Delta\psi_1 + \Delta\psi_2 + \Delta\psi_3 + \cdots + \Delta\psi_n \\ &= B_1 \cos\alpha_1 \, \Delta S_1 + B_2 \cos\alpha_2 \, \Delta S_2 + B_3 \cos\alpha_3 \, \Delta S_3 + \cdots \\ &\quad + B_n \cos\alpha_n \, \Delta S_n \\ &= \sum_{j=1}^{n} B_j \cos\alpha_j \, \Delta S_j \\ &= \sum_{j=1}^{n} B_j (\Delta S_j) \cos\alpha_j\end{aligned}$$

$$[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta S_j \, \mathbf{a}_{nj} \qquad \boxed{[\psi]_S = \sum_{j=1}^{n} \mathbf{B}_j \cdot \Delta \mathbf{S}_j}$$

For a numerical example, let us consider the magnetic field given by

$$\mathbf{B} = 3xy^2\mathbf{a}_z \text{ Wb/m}^2$$



$$\mathbf{B}_{ij} = 3(2i - 1)(2j - 1)^2 0.001\mathbf{a}_z$$

$$\Delta\mathbf{S}_{ij} = 0.04\,\mathbf{a}_z \qquad \text{for all } i \text{ and } j$$

$$
\begin{aligned}
[\psi]_S &= \sum_{i=1}^{5}\sum_{j=1}^{5}\mathbf{B}_{ij}\cdot\Delta\mathbf{S}_{ij} \\
&= \sum_{i=1}^{5}\sum_{j=1}^{5}3(2i-1)(2j-1)^2 0.001\mathbf{a}_z\cdot 0.04\mathbf{a}_z \\
&= 0.00012\sum_{i=1}^{5}\sum_{j=1}^{5}(2i-1)(2j-1)^2 \\
&= 0.00012(1+3+5+7+9)(1+9+25+49+81) \\
&= 0.495 \text{ Wb}
\end{aligned}
$$

$$
\boxed{[\psi]_S = \int_S \mathbf{B}\cdot d\mathbf{S}}
$$

$$
d\mathbf{S} = dx\,dy\,\mathbf{a}_z
$$

The value of $\mathbf{B}\cdot d\mathbf{S}$ at the point is

$$
\begin{aligned}
\mathbf{B}\cdot d\mathbf{S} &= 3xy^2\,\mathbf{a}_z\cdot dx\,dy\,\mathbf{a}_z \\
&= 3xy^2\,dx\,dy
\end{aligned}
$$

Thus, the required magnetic flux is given by

$$
\begin{aligned}
[\psi]_S &= \int_S \mathbf{B}\cdot d\mathbf{S} \\
&= \int_{x=0}^{1}\int_{y=0}^{1} 3xy^2\,dx\,dy = 0.5 \text{ Wb}
\end{aligned}
$$

Example 2.2 Evaluation of a closed surface integral

Let us consider the magnetic field

$$\mathbf{B} = (x + 2)\mathbf{a}_x + (1 - 3y)\mathbf{a}_y + 2z\mathbf{a}_z$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_{abcd} \mathbf{B} \cdot d\mathbf{S} + \int_{efgh} \mathbf{B} \cdot d\mathbf{S} + \int_{adhe} \mathbf{B} \cdot d\mathbf{S} + \int_{bcgf} \mathbf{B} \cdot d\mathbf{S} + \int_{aefb} \mathbf{B} \cdot d\mathbf{S} + \int_{dhgc} \mathbf{B} \cdot d\mathbf{S}$$

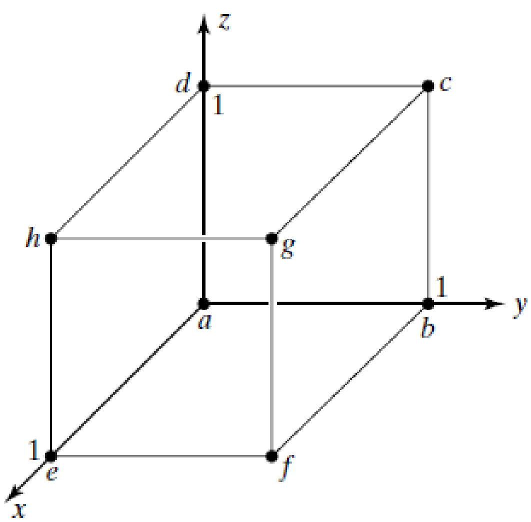


FIGURE 2.9

For evaluating the surface integral of a vector field over a closed surface.

Finally,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = -2 + 3 - 1 - 2 + 0 + 2 = 0$$

D2.3. Given $\mathbf{B} = (y\mathbf{a}_x - x\mathbf{a}_y)$ Wb/m^2, find by evaluating $\mathbf{B} \cdot \Delta\mathbf{S}$ the approximate absolute value of the magnetic flux crossing from one side to the other side of an infinitesimal surface of area 0.001 m^2 at the point (1, 2, 1) for each of the following orientations of the surface: **(a)** in the $x = 1$ plane; **(b)** on the surface $2x^2 + y^2 = 6$; and **(c)** normal to the unit vector $\frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)$.

Ans. **(a)** 2×10^{-3} Wb; **(b)** $(1/\sqrt{2}) \times 10^{-3}$ Wb; **(c)** 10^{-3} Wb.

D2.4. For the vector field $\mathbf{A} = x(\mathbf{a}_x + \mathbf{a}_y)$, find the absolute value of $\int \mathbf{A} \cdot d\mathbf{S}$ over the following plane surfaces: **(a)** square having the vertices at (0, 0, 0), (0, 2, 0), (0, 2, 2), and (0, 0, 2); **(b)** square having the vertices at (2, 0, 0), (2, 2, 0), (2, 2, 2), and (2, 0, 2); **(c)** square having the vertices at (0, 0, 0), (2, 0, 0), (2, 0, 2), and (0, 0, 2); and **(d)** triangle having the vertices at (0, 0, 0), (2, 0, 0), and (0, 0, 2).

Ans. **(a)** 0; **(b)** 8; **(c)** 4; **(d)** $\frac{4}{3}$.