Session 9

Maxwell's Equations in Differential Forms

FARADAY'S LAW AND AMPÈRE'S CIRCUITAL LAW

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\mathbf{E} = E_x(z, t)\mathbf{a}_x$$

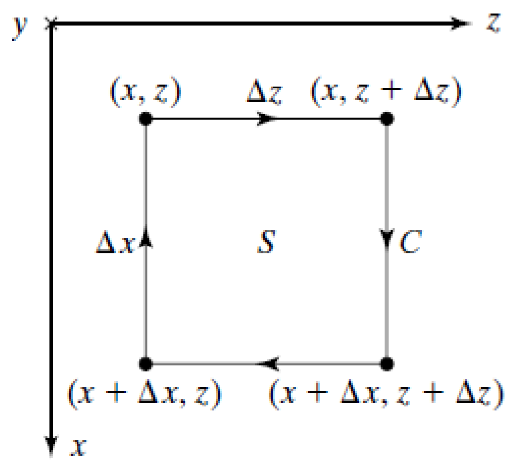


FIGURE 3.1

Infinitesimal rectangular path lying in a plane parallel to the *xz*-plane.

$$\int_{(x,\, z)}^{(x,\, z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{since } E_z = 0$$

$$\int_{(x,\, z+\Delta z)}^{(x+\Delta x,\, z+\Delta z)} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{z+\Delta z}\, \Delta x$$

$$\int_{(x+\Delta x,\, z+\Delta z)}^{(x+\Delta x,\, z)} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{since } E_z = 0$$

$$\int_{(x+\Delta x,\, z)}^{(x,\, z)} \mathbf{E} \cdot d\mathbf{l} = -[E_x]_z \, \Delta x$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = [E_x]_{z+\Delta z} \, \Delta x - [E_x]_z \, \Delta x$$

$$= \{[E_x]_{z+\Delta z} - [E_x]_z\} \, \Delta x$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,\, z)} \, \Delta x \, \Delta z$$

$$\frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\frac{\partial [B_y]_{(x,\, z)}}{\partial t}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{[E_x]_{z+\Delta z} - [E_x]_z}{\Delta z} = -\lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}} \frac{\partial [B_y]_{(x,\, z)}}{\partial t}$$

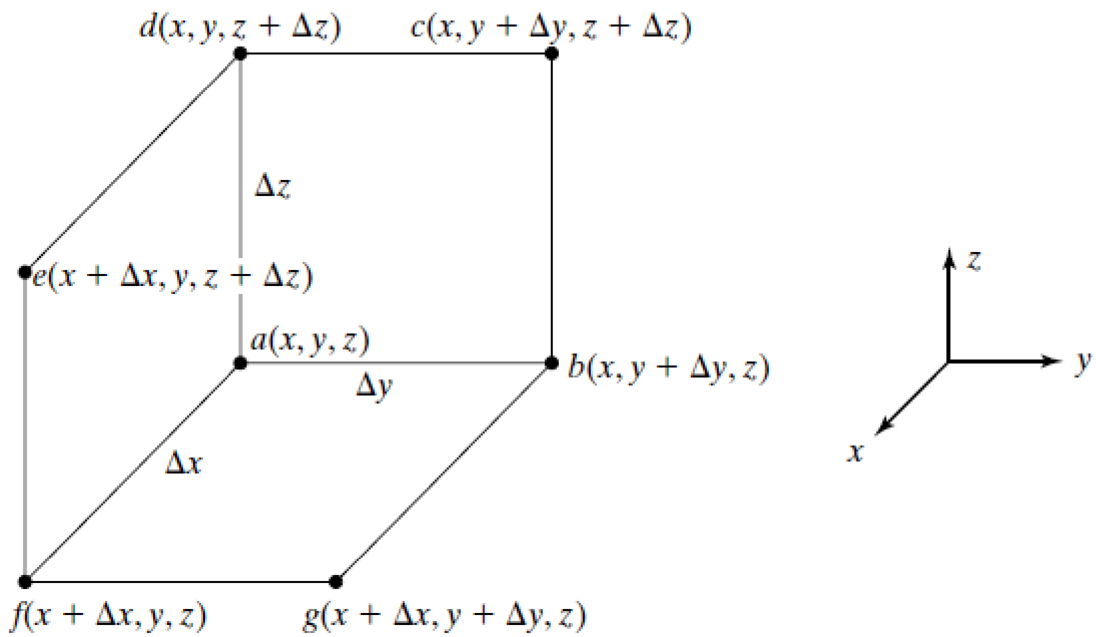
$$\boxed{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}}$$

$$\boxed{\mathbf{E} = E_x(x, y, z, t)\mathbf{a}_x + E_y(x, y, z, t)\mathbf{a}_y + E_z(x, y, z, t)\mathbf{a}_z}$$

$$\oint_{abcda} \mathbf{E} \cdot d\mathbf{l} = [E_y]_{(x,\,z)}\,\Delta y + [E_z]_{(x,y+\Delta y)}\,\Delta z - [E_y]_{(x,\,z+\Delta z)}\,\Delta y - [E_z]_{(x,\,y)}\,\Delta z$$

$$\oint_{adefa} \mathbf{E} \cdot d\mathbf{l} = [E_z]_{(x,\,y)}\,\Delta z + [E_x]_{(y,\,z+\Delta z)}\,\Delta x - [E_z]_{(x+\Delta x,\,y)}\,\Delta z - [E_x]_{(y,\,z)}\,\Delta x$$

$$\oint_{afgba} \mathbf{E} \cdot d\mathbf{l} = [E_x]_{(y,\,z)}\,\Delta x + [E_y]_{(x+\Delta x,\,z)}\,\Delta y - [E_x]_{(y+\Delta y,\,z)}\,\Delta x - [E_y]_{(x,\,z)}\,\Delta y$$



$$\int_{abcd} \mathbf{B} \cdot d\mathbf{S} = [B_x]_{(x,\,y,\,z)}\, \Delta y\, \Delta z$$

$$\int_{adef} \mathbf{B} \cdot d\mathbf{S} = [B_y]_{(x,\,y,\,z)}\, \Delta z\, \Delta x$$

$$\int_{afgb} \mathbf{B} \cdot d\mathbf{S} = [B_z]_{(x,\,y,\,z)}\, \Delta x\, \Delta y$$

$$\frac{[E_z]_{(x,\,y+\Delta y)} - [E_z]_{(x,\,y)}}{\Delta y} - \frac{[E_y]_{(x,\,z+\Delta z)} - [E_y]_{(x,\,z)}}{\Delta z} = -\frac{\partial [B_x]_{(x,\,y,\,z)}}{\partial t}$$

$$\frac{[E_x]_{(y,\,z+\Delta z)} - [E_x]_{(y,\,z)}}{\Delta z} - \frac{[E_z]_{(x+\Delta x,\,y)} - [E_z]_{(x,\,y)}}{\Delta x} = -\frac{\partial [B_y]_{(x,\,y,\,z)}}{\partial t}$$

$$\frac{[E_y]_{(x+\Delta x,\,z)} - [E_y]_{(x,\,z)}}{\Delta x} - \frac{[E_x]_{(y+\Delta y,\,z)} - [E_x]_{(y,\,z)}}{\Delta y} = -\frac{\partial [B_z]_{(x,\,y,\,z)}}{\partial t}$$

$$\boxed{\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t} \end{aligned}}$$

$$\boxed{\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t}}$$

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{a}_z$$

$$= -\frac{\partial B_x}{\partial t}\mathbf{a}_x - \frac{\partial B_y}{\partial t}\mathbf{a}_y - \frac{\partial B_z}{\partial t}\mathbf{a}_z$$

$$\boxed{\nabla = \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{3.17}$$

Equation (3.17) is Maxwell's equation in differential form corresponding to Faraday's law. It tells us that at a point in an electromagnetic field, the curl of the electric field intensity is equal to the time rate of decrease of the magnetic flux density. We shall discuss curl further in Section 3.3, but note that for static fields, $\nabla \times \mathbf{E}$ is equal to the null vector. Thus, for a static vector field to be realized as an electric field, the components of its curl must all be zero.

CARTESIAN

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

CYLINDRICAL

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\mathbf{a}_r}{r} & \mathbf{a}_\phi & \frac{\mathbf{a}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

SPHERICAL

$$\nabla \times \mathbf{A} = \begin{vmatrix} \frac{\mathbf{a}_r}{r^2 \sin\theta} & \frac{\mathbf{a}_\theta}{r\sin\theta} & \frac{\mathbf{a}_\phi}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\mathbf{B})$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t}(\mathbf{D})$$

$$\mathbf{H} = H_x(x, y, z, t)\mathbf{a}_x + H_y(x, y, z, t)\mathbf{a}_y + H_z(x, y, z, t)\mathbf{a}_z$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_z & H_y & H_z \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Example 3.3 Simultaneous satisfaction of Faraday's and Ampere's circuital laws by E and B

Given $\mathbf{E} = E_0 z^2 e^{-t} \mathbf{a}_x$ in free space ($\mathbf{J} = \mathbf{0}$). We wish to determine if there exists a magnetic field such that both Faraday's law and Ampère's circuital law are satisfied simultaneously.

Using Faraday's law and Ampère's circuital law in succession, we have

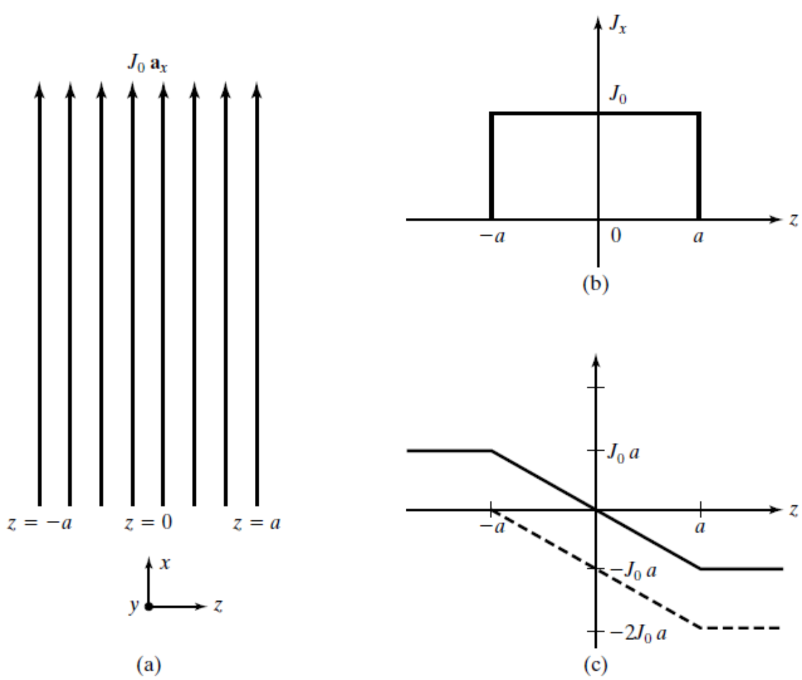
$$\frac{\partial B_y}{\partial t} = -\frac{\partial E_x}{\partial z} = -2E_0 z e^{-t}$$
$$B_y = 2E_0 z e^{-t}$$
$$H_y = \frac{2E_0}{\mu_0} z e^{-t}$$
$$\frac{\partial D_x}{\partial t} = -\frac{\partial H_y}{\partial z} = -\frac{2E_0}{\mu_0} e^{-t}$$
$$D_x = \frac{2E_0}{\mu_0} e^{-t}$$
$$E_x = \frac{2E_0}{\mu_0 \varepsilon_0} e^{-t}$$
$$\mathbf{E} = \frac{2E_0}{\mu_0 \varepsilon_0} e^{-t} \mathbf{a}_x$$

which is not the same as the original **E**. Hence, a magnetic field does not exist which together with the given **E** satisfies both laws simultaneously. The pair of fields $\mathbf{E} = E_0 z^2 e^{-t} \mathbf{a}_x$ and $\mathbf{B} = 2E_0 z e^{-t} \mathbf{a}_y$ satisfies only Faraday's law, whereas the pair of fields $\mathbf{B} = 2E_0 z e^{-t} \mathbf{a}_y$ and $\mathbf{E} = (2E_0/\mu_0 \varepsilon_0) e^{-t} \mathbf{a}_x$ satisfies only Ampère's circuital law.

FIGURE 3.4

The determination of magnetic field due to a current distribution.

Example 3.4 Magnetic field of a current distribution from Ampere's circuital law in differential form



tic field everywhere.

FIGURE 3.4

The determination of magnetic field due to a current distribution.

$$\frac{\partial H_y}{\partial z} = -J_x$$

Integrating both sides with respect to z, we obtain

$$H_y = -\int_{-\infty}^{z} J_x \, dz + C$$

where C is the constant of integration.

The variation of J_x with z is shown in Fig. 3.4(b). Integrating $-J_x$ with respect to z, that is, finding the area under the curve of Fig. 3.4(b) as a function of z, and taking its negative, we obtain the result shown by the dashed curve in Fig. 3.4(c) for $-\int_{-\infty}^{z} J_x \, dz$. From symmetry considerations, the field must be equal and opposite on either side of the current region $-a < z < a$. Hence, we choose the constant of integration C to be equal to $J_0 a$, thereby obtaining the final result for H_y as shown by the solid curve in Fig. 3.4(c). Thus, the magnetic field intensity due to the current distribution is given by

$$\mathbf{H} = \begin{cases} J_0 a \mathbf{a}_y & \text{for } z < -a \\ -J_0 z \mathbf{a}_y & \text{for } -a < z < a \\ -J_0 a \mathbf{a}_y & \text{for } z > a \end{cases}$$

The magnetic flux density, $\mathbf{B}$, is equal to $\mu_0 \mathbf{H}$.

GAUSS' LAWS AND THE CONTINUITY EQUATION

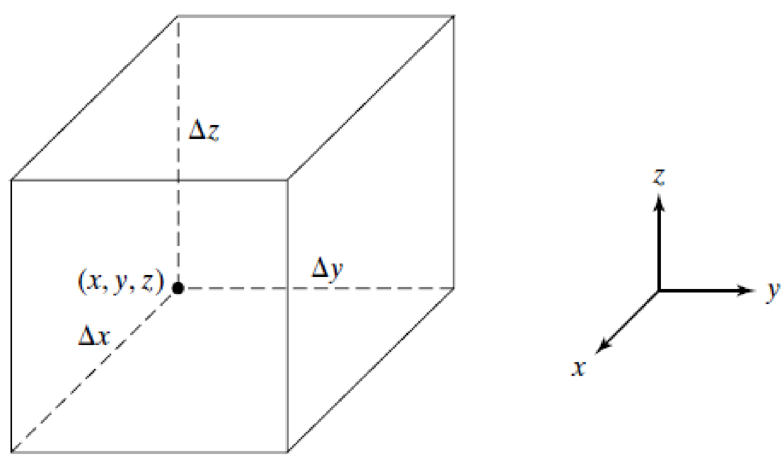


FIGURE 3.5

Infinitesimal rectangular box.

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dv}$$

$$\boxed{\mathbf{D} = \mathbf{D}_x(x, y, z, t)\mathbf{a}_x + D_y(x, y, z, t)\mathbf{a}_y + D_z(x, y, z, t)\mathbf{a}_z}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_x]_x \, \Delta y \, \Delta z \qquad \text{for the surface } x = x \tag{3.28a}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_x]_{x+\Delta x} \, \Delta y \, \Delta z \qquad \text{for the surface } x = x + \Delta x \tag{3.28b}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_y]_y \, \Delta z \, \Delta x \qquad \text{for the surface } y = y \tag{3.28c}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_y]_{y+\Delta y} \, \Delta z \, \Delta x \qquad \text{for the surface } y = y + \Delta y \tag{3.28d}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = -[D_z]_z \, \Delta x \, \Delta y \qquad \text{for the surface } z = z \tag{3.28e}$$

$$\int \mathbf{D} \cdot d\mathbf{S} = [D_z]_{z+\Delta z} \, \Delta x \, \Delta y \qquad \text{for the surface } z = z + \Delta z \tag{3.28f}$$

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{S} = {} & \{[D_x]_{x+\Delta x} - [D_x]_x\} \, \Delta y \, \Delta z \\ & + \{[D_y]_{y+\Delta y} - [D_y]_y\} \, \Delta z \, \Delta x \\ & + \{[D_z]_{z+\Delta z} - [D_z]_z\} \, \Delta x \, \Delta y\end{aligned}$$

$$\int_V \rho \, dv = \rho(x, y, z, t) \cdot \Delta x \, \Delta y \, \Delta z = \rho \Delta x \, \Delta y \, \Delta z$$

$$\frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} + \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \rho$$

$$\lim_{\Delta x \to 0} \frac{[D_x]_{x+\Delta x} - [D_x]_x}{\Delta x} + \lim_{\Delta y \to 0} \frac{[D_y]_{y+\Delta y} - [D_y]_y}{\Delta y} + \lim_{\Delta z \to 0} \frac{[D_z]_{z+\Delta z} - [D_z]_z}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \rho$$

$$\boxed{\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho}$$

$$\left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot (D_x \mathbf{a}_x + D_y \mathbf{a}_y + D_z \mathbf{a}_z) = \rho$$

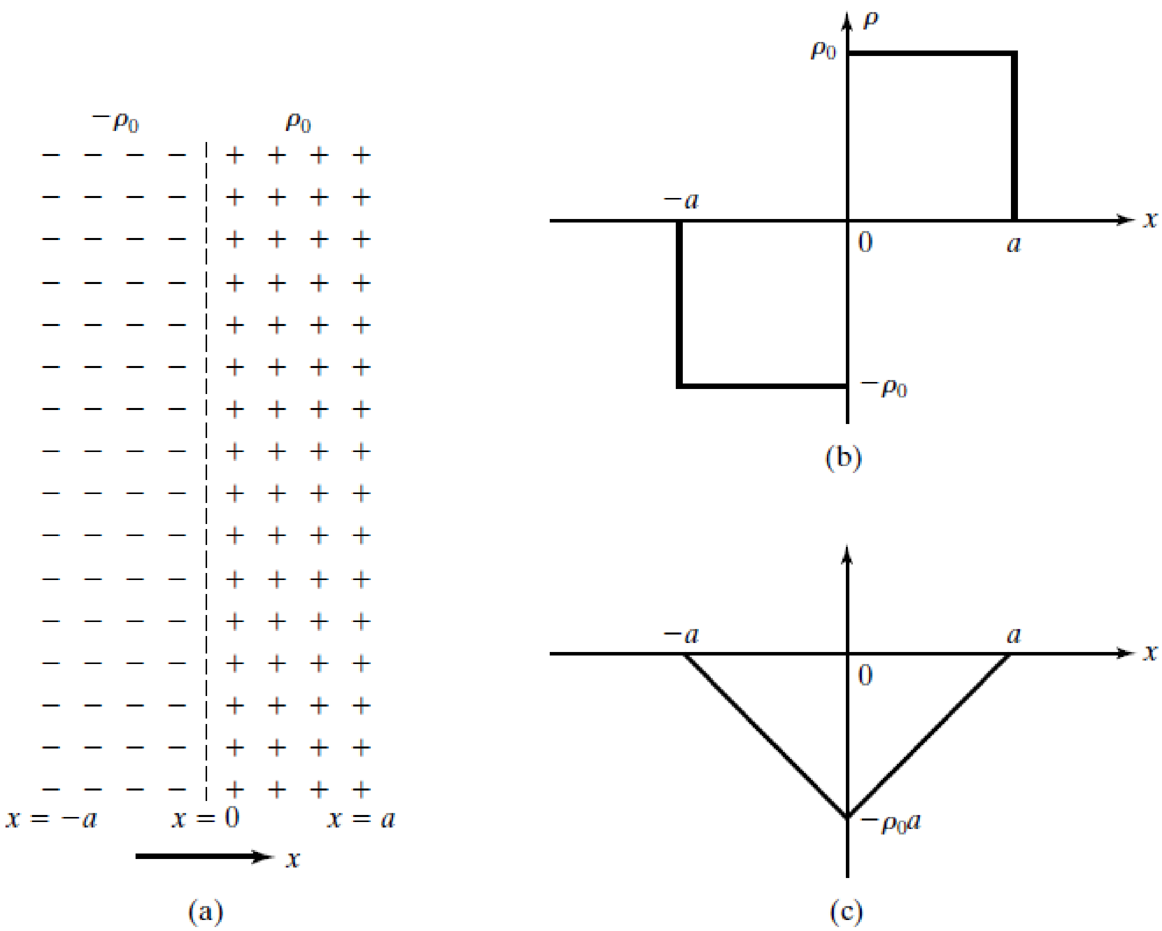
$$\boxed{\nabla \cdot \mathbf{D} = \rho}$$

Example 3.5 Electric field of a charge distribution from Gauss' law in differential form

Let us consider the charge distribution given by

$$\rho = \begin{cases} -\rho_0 & \text{for } -a < x < 0 \\ \rho_0 & \text{for } 0 < x < a \end{cases}$$

as shown in Fig. 3.7(a), where ρ_0 is a constant, and find the electric field everywhere.



Since the charge density is independent of y and z, the field is also independent of y and z, thereby giving us $\partial D_y/\partial y = \partial D_z/\partial z = 0$ and reducing Gauss' law for the electric field to

$$\frac{\partial D_x}{\partial x} = \rho$$

$$D_x = \int_{-\infty}^{x} \rho \, dx + C \qquad \mathbf{D} = \begin{cases} 0 & \text{for } x < -a \\ -\rho_0(x + a)\mathbf{a}_x & \text{for } -a < x < 0 \\ \rho_0(x - a)\mathbf{a}_x & \text{for } 0 < x < a \\ 0 & \text{for } x > a \end{cases}$$

The electric field intensity, $\mathbf{E}$, is equal to $\mathbf{D}/\varepsilon_0$.

Cartesian

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Spherical

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0}$$

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 = \int_V 0 \, dv \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

$$\boxed{\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt}\int_V \rho \, dv}$$

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}}$$