Sampling & Quantization

Discrete signals:

Sampled in Time and Quantized in Amplitude

How to minimize the effects?

Analog → Sample → Quantize → Encode → *Digital*

History of sampling/quantization Borel 1897…

Whittaker 1915 (son 1929), Nyquist 1928, Kotelnikov 1933, Shannon 1949, Linvill 1949 Widrow 1952

Sampling is needed to study the quantization

Sampling theory in a very useful form was developed by Linvill in his doctoral dissertation at MIT, 1949

Frequency domain description of sampling

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The Model for Sampling Process

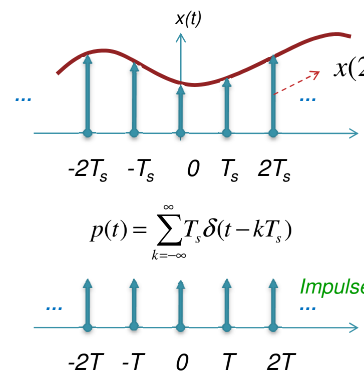
$$x_s(t) = x(t)\,p(t)$$

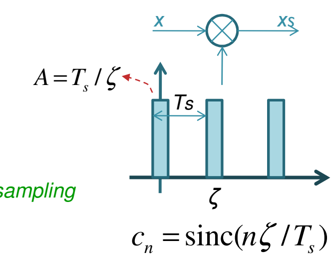
$p(t)$ is a periodic function, period = T_s

$$p(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j2\pi nt/T_s}$$

$$X_s(f) = \sum_{n=-\infty}^{+\infty} c_n X(f - nf_s)$$

$$x[k] = x(kT_s)$$





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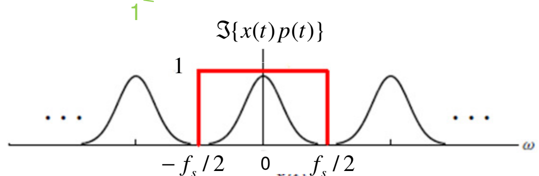
The Reconstruction Model

LP sampling Theorem, Nyquist Rate, Aliasing

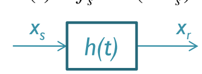
$$x_r(t) = [x(t)p(t)] * h(t) = \sum_{k=-\infty}^{+\infty} T_s x(kT_s) h(t - kT_s)$$

$$Ideal:\quad x_r(t) = f_s T_s \sum_{k=-\infty}^{+\infty} x(kT_s)\,\mathrm{sinc}(f_s(t - kT_s))$$

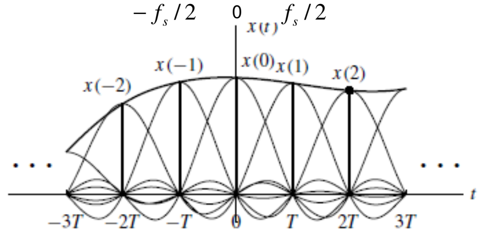
1



$h(t) = f_s\,\mathrm{sinc}(t/T_s)$



$$X_s(f) = \sum_{n=-\infty}^{+\infty} X(f - nf_s)$$



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Extending to Random Signals

Definitions: *Mean, Auto-correlation, Auto-covariance*

$$m_s(t) = E[s(t)]$$

$$R_s(t_1,t_2) = E[s(t_1)s^*(t_2)]$$

$$C_s(t_1,t_2) = E[(s(t_1) - E[s(t_1)])(s(t_2) - E[s(t_2)])^*]$$
$$= R_s(t_1,t_2) - m_s(t_1)m_s^*(t_2)$$

Cross-correlation, Cross-covariance

$$R_{s_1,s_2}(t_1,t_2) = E[s_1(t_1)s_2^*(t_2)]$$

$$C_{s_1,s_2}(t_1,t_2) = E[(s_1(t_1) - E[s_1(t_1)])(s_2(t_2) - E[s_2(t_2)])^*]$$
$$= R_{s_1,s_2}(t_1,t_2) - m_{s_1}(t_1)m_{s_2}^*(t_2)$$

Extending to Random Process…

Stationary: *A random process s(t) is said to be stationary if it is statistically indistinguishable from its delayed version.*
s(t) and s(t+d) have the same statistical properties

$$\forall d, t_1, t_2, t \in \Re:$$
$$m_s(t) = m_s(t-d) = m_s(0)$$
$$R_s(t_1,t_2) = R_s(t_1-d, t_2-d) = R_s(t_1-t_2, 0) \quad \dots$$

Normally a weaker property is used which is easier to verify
Wide Sense Stationary (WSS)*: Only first and second moments*

$$R_s(t_1,t_2) = R_s(t_1-t_2)$$
$$= R_s(\tau) = E[s(t)s^*(t-\tau)], \quad \tau = t_1 - t_2$$

Extending to Random Process…

Power Spectral Density (PSD):

$$S_s(f) = \Im\{R_s(\tau)\}$$

Can be stated as a definition…
*Or as **Wiener–Khinchin** theorem Starting from the following definition and proving the above formula*

$$S_s(f) = \lim_{T\to\infty}\left[\frac{|\Im\{s_T(t)\}|^2}{T}\right] = E\left[|\Im\{s(t)\}|^2\right]$$

Ergodicity: *A stationary random process s is ergodic if time averages along a realization equal statistical averages across realizations.*
For WSS processes, we are primarily interested in ergodicity for the mean and autocorrelation functions.

Extending to Random Process…

Cyclostationary Random process: *The random process s(t) is cyclostationary with respect to time interval T if it is statistically indistinguishable from s(t–kT) for any integer k.*

As with the concept of stationary, relax the notion of cyclostationary by considering only the first and second order statistics.

Wide Sense Cyclostationary Random process:

$$\forall t_1, t_2, t \in \Re, \ \exists T \in \Re:$$
$$m_s(t) = m_s(t-T)$$
$$R_s(t_1,t_2) = R_s(t_1-T, t_2-T)$$

Extending to Random Process…

Stationarizing a cyclostationary process
Theorem: *Let s(t) be a cyclostationary random process with respect to the time interval T and D is a uniformly distributed random variable over [0,T], independent of s(t). Then s(t–D) is a stationary random process. Similarly, if s(t) is wide sense cyclostationary, then s(t–D) ("stationarized" version of s(t)) is a WSS random process.*

Proof outline: For any delay a , a+D=kT+d, where d~U[0,T] k is an integer

$$v(t) := s(t-D), \qquad \tilde{s}(t) := s(t-kT) \sim s(t)$$
$$v(t-a) \sim \tilde{s}(t-d)$$

So v(t-a) is indistinguishable from v(t) for any a → v is a stationary process

Extending to Random Process…

Moreover:

$$m_v(t) = \frac{1}{T}\int_0^T E[v|D]dD = \frac{1}{T}\int_0^T m_s(D)dD$$

$$R_v(t,t+\tau) = \frac{1}{T}\int_0^T R_s(t-D, t+\tau-D \mid D)dD$$

PSD for Cyclostationary Processes

For wide sense cyclostationary $s(t)$ with respect to time interval T, the PSD is defined as:

$$S_s(f) = \Im\{R_s(\tau)\}$$

$$R_s(\tau) = E[s(t-D)s^*(t-D-\tau)]$$

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Low-pass Sampling of Random Signals

$$X_s(t) = X(t)P(t)$$

$$P(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT_s-D), \quad D \sim U\,[0,T_s]$$

Theorem: *Assuming x is stationary (at least in a wide sense), D is used to avoid having cyclo-stationary signals*
Using PSD (Power Spectral Density) formulation

$$R_{X_s}(\tau) = E[X_s(t)X^*{}_s(t+\tau)]$$

$$S_{X_s}(f) = \Im(R_{X_s}(\tau)) = f_s^2 \sum_{n=-\infty}^{+\infty} S_X(f-nf_s)$$

Same Results

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Low-pass Sampling of Random Signals

Proof outline: For a fix D we can write:

$$P(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT_s-D) \overset{F.S.}{\rightarrow} = f_s \sum_{n=-\infty}^{+\infty} e^{j2\pi nt/T_s} e^{-j2\pi nD/T_s}$$

$$S_{X_s}(f) = \int_0^{T_s} E[\Im(X_s(t))\Im(X_s(t))^* \mid D] f(D)dD$$

$$= \int_0^{T_s} f_s^2 E[\sum_{n=-\infty}^{+\infty} X(f-nf_s)e^{-j2\pi nD/T_s} \ldots$$

$$\ldots \sum_{m=-\infty}^{+\infty} X^*(f-mf_s)e^{j2\pi mD/T_s} f(D)dD]$$

$$= f_s^2 \sum_{n=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} E[X(f-nf_s)X^*(f-mf_s)]\left\{\frac{1}{T_s}\int_0^{T_s} e^{j2\pi(m-n)D/T_s}dD\right\}$$

$$= f_s^2 \sum_{n=-\infty}^{+\infty} S_X(f-nf_s)$$

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Band-pass Sampling

Band-pass Sampling Theorem for Real Band-pass signals

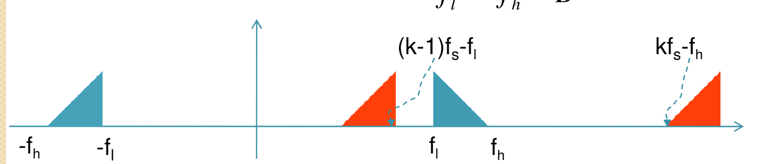
Not all frequencies > f_s^* are usable
Until $f_s > 2f_h$ which is the low pass theorem bound

$$Bandwidth \;\; = B$$

$$Highest \;\; frequency \;\; = f_h$$

$$m = [f_h / B] \rightarrow f_s^* = 2f_h / m$$

$$f_l = f_h - B$$



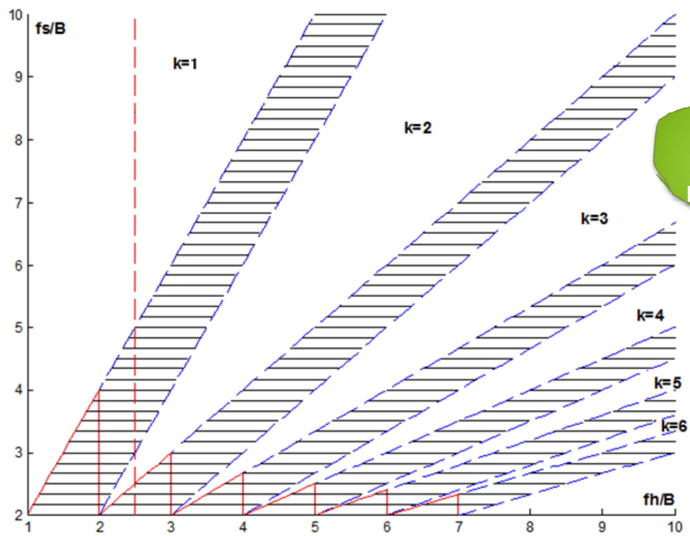
$$(k-1)f_s - f_l \le f_l \Rightarrow f_h / B - 1 \ge ((k-1)f_s / 2B$$

$$kf_s - f_h \ge f_h \Rightarrow f_h / B \le kf_s / 2B$$

$$\frac{2}{k}(f_h / B) \le f_s / B \le \frac{2}{k-1}(f_h / B - 1)$$

Band-pass Sampling…

Shaded regions, constraints not satisfied $f_h >> B \Rightarrow f_s \approx 2B$



ACT I
Draw in
MATLAB

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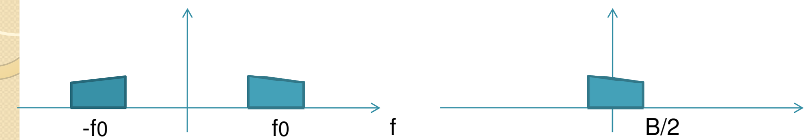
Bandpass Signals, LP Representation, Complex Envelope

2 signals with B/2 Band width, fs≥2B



$$x(t) = A(t)\cos(2\pi f_0 t + \varphi(t))$$
$$= x_i(t)\cos(2\pi f_0 t) + x_q(t)\sin(2\pi f_0 t)$$
$$x_i(t) = A(t)\cos\varphi(t)$$
$$x_q(t) = -A(t)\sin\varphi(t)$$
$$\tilde{x}(t) = x_i(t) + jx_q(t)$$
$$\tilde{X}(f) = \tilde{X}_i(f) + j\tilde{X}_q(f)$$

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Quantization, Definition, Basic form

Occurs whenever physical quantities are represented numerically
In a Computer: Operations or Memory assignments

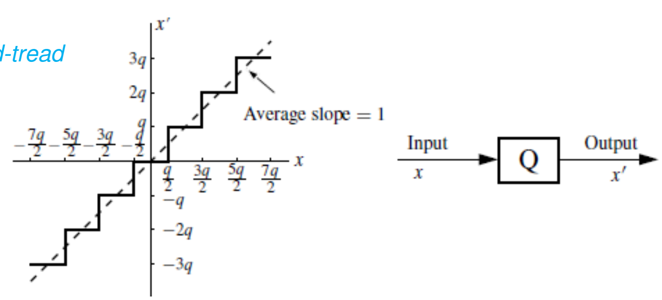
Statistical and non-linear in nature

Q: quantization operator

q: quantization step

x-x' : quantization noise

Mid-tread



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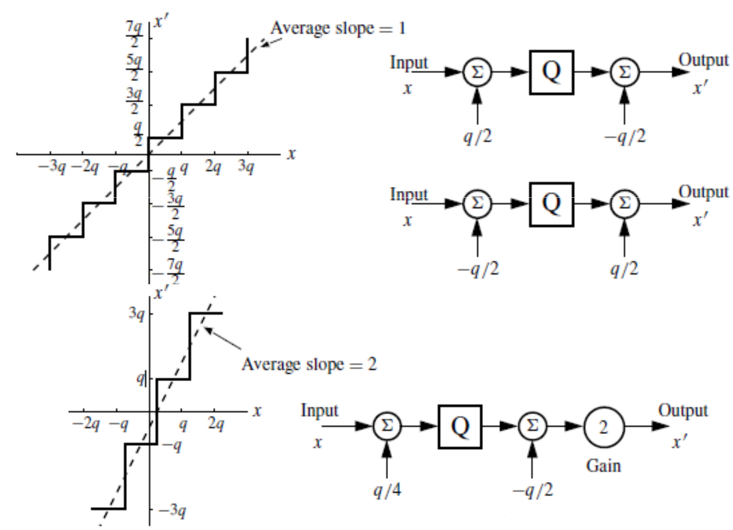
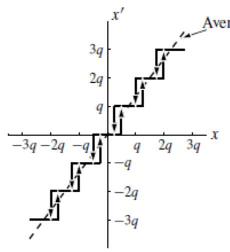
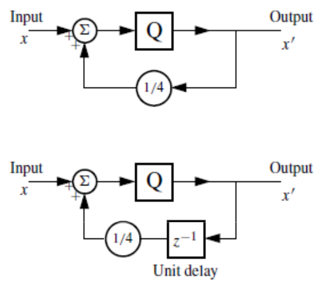
Quantization…

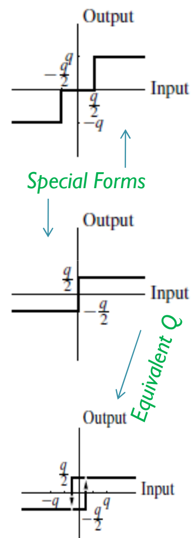
Mid-riser and Scaled



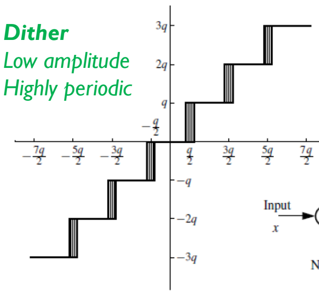
Hysteresis, Saturation, Dithering

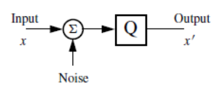




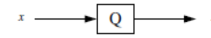


Dither
Low amplitude
Highly periodic





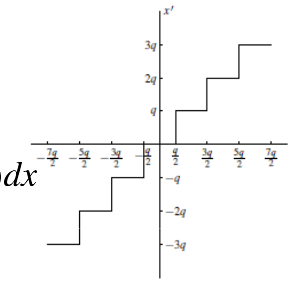
Analogy between Sampling and Quantization



$$-q/2 \le x \le q/2 \quad \Rightarrow \quad x' = 0$$

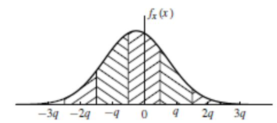
$$p(x'=0) = \int_{-q/2}^{q/2} f_X(x)dx$$

$$f_{X'}(x') = \sum_{m=-\infty}^{+\infty} \delta(x'-mq)\int_{mq-q/2}^{mq+q/2} f_X(x)dx$$

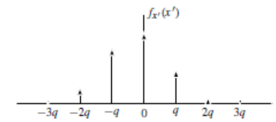


Looks like a sampling process!
A bit different : Area Sampling…

$$f_N(n) = \begin{cases} 1/q, & -q/2 \le n \le q/2 \\ 0\ , & ow \end{cases}$$



$$p(x) = \sum_{m=-\infty}^{+\infty} q\,\delta(x-mq)$$



$$f_{X'}(x) = (f_X(x) * f_N(x)) \times p(x)$$

Quantization in CF domain

$$\phi_X(\omega) = \int_{-\infty}^{+\infty} f_X(x)e^{j\omega x}dx = E(e^{j\omega X})$$

$$\phi_N(\omega) = \int_{-q/2}^{+q/2} (1/q)e^{j\omega n}dn = \text{sinc}(q\omega/2\pi)$$

CF of x' in terms of x

$$\phi_{X'}(\omega) = \sum_{l=-\infty}^{+\infty} \phi_X(\omega + l\omega_s)\,\text{sinc}(q(\omega + l\omega_s)/2\pi),\ \omega_s = 2\pi/q$$

A repeating spectrum pattern …
Different from additive model of quantization noise

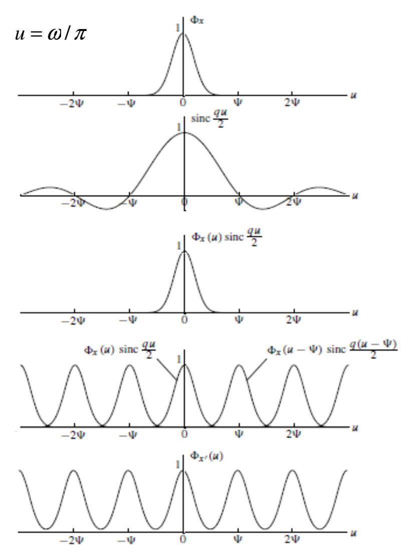
$$f_X(v) * f_N(v) = f_{X+N}(v), \quad if\ \ X\ \ and\ \ N\ \ are\ independent$$

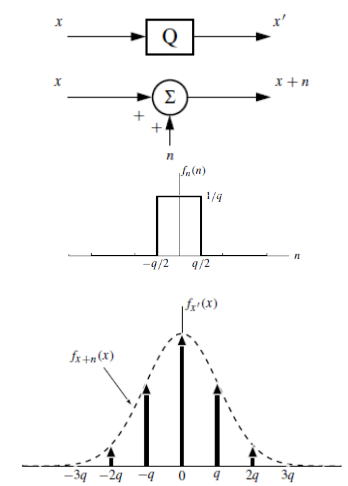
$$f_{X'}(v) = f_{X+N}(v) * p(v)$$

N, zero mean, variance $q^2/12$

X+N is not X'!

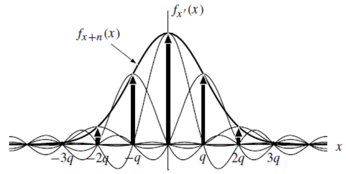
$u = \omega/\pi$





Quantization Theorems

Widrow 1952



- **Theorem I**

If CF of x is band-limited, CF and PDF of x can be derived from the CF and PDF of x'.

$$\phi_X(\omega) = 0 \quad for\ |\omega| > \pi / q$$

- **Theorem II (weaker)**

(overlap occurs but not enough to ruin the CF derivatives at zero)

If CF of x is band-limited, moments of x can be derived from the moments of x'.

$$\phi_X(\omega) = 0 \quad for\ |\omega| > 2\pi / q$$

Adding mean to x still keeps these theorems applicable:

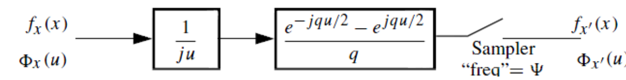
$$\phi_{X+\mu}(\omega) = e^{j\mu\omega}\phi_X(\omega)$$

Quantization Theorems…

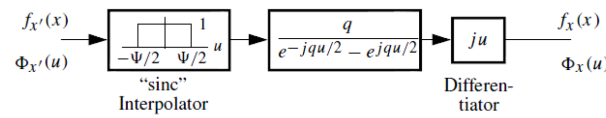
pdf Reconstruction

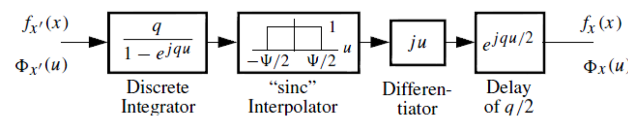
$$f_{X+N}(x) = f_X(x) * f_N(x) = \frac{1}{q}\{F(x+q/2) - F(x-q/2)\}$$

Sampling pdf of x+n yields pdf of x'



if conditions of Thrm I satisfied sampling can be reversed by sinc interpolator





Quantization Theorems…

When Conditions of Thrm II are satisfied

$$E(X') = E(X+N) = E(X)$$

$$E(X'^2) = E(X^2) + q^2/12$$

$$E(X'^3) = E(X^3) + (1/4)E(X)q^2$$

$$E(X'^4) = E(X^4) + (1/2)E(X^2)q^2 + (1/80)q^4$$

…

$$E(X'^r) = E(X^r) + M_r$$

X' moments can be calculated from quantized data → moments of X, solution to …

Sheppard formulas → Possible if QT I or II are satisfied

$$E(X) = E(X')$$

$$E(X^2) = E(X'^2) - q^2/12$$

$$E(X^3) = E(X'^3) - (1/4)E(X')q^2$$

…

Quantization Theorems…

When moments are in hands

$$\phi_X(\omega) = E(e^{j\omega X}) = \int_{-\infty}^{+\infty} f_X(x)e^{j\omega x}dx$$

$$\phi_X(0) = 1$$

$$\phi_X'(0) = jE(X) \qquad \rightarrow \phi_X(\omega) \overset{?}{=} \sum_{n=0}^{\infty} \frac{j^n}{n!}E(X^n)\omega^n$$

…

$$\phi_X^{(n)}(0) = (j)^n E(X^n)$$

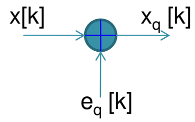
Does Taylor / Maclaurin series around 0 converge?

- Band limited CFs are not analytic and Cannot be expressed in Maclaurin series around 0 → Still QT I works!
- When QT II is satisfied and QT I is not, moments are there but we cannot have CF. → Still partly possible for approximately band limited CFs (must be checked!).

PQN, Pseudo Quantization Noise model

Fixed Point: quantization levels = 2^b

$$(SNR)_q = \frac{E\{x^2[k]\}}{E\{e_q^{\,2}[k]\}} = \frac{S}{N_q} \qquad |e_q[k]| \le q/2$$



If 'n' is large enough and x varies significantly

from sample to sample then e ~ U[-q/2,q/2]

$$E\{e_q[k]\} = 0 \qquad N_q = E\{e_q^{\,2}[k]\} = q^2/12$$

$$Dynamic\ \ Range : D = 2^b q \qquad Signal\ \ Power : S$$

$$Crest\ \ Factor : CF = \frac{\sqrt{S}}{D/2}$$

$$SNR_q = 3CF^2 2^{2b}$$

$$SNR_q(dB) = 4.7712 + 20\log_{10} CF + 6.0206b$$