Postprocessing

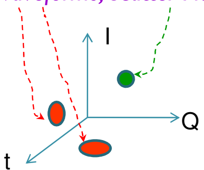
Manipulating the simulation generated data into useful forms

Graphics, Formatting, Estimation, Statistical analysis, …

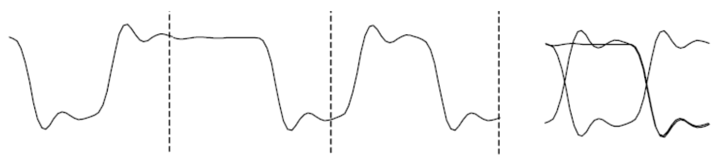
Postprocessors task can be process extensive, graphic extensive or simple

Graphics

Waveforms, Scatter Plot, Eye Diagram, Constellation



Eye Diagram: No quantitative info, Sweeping in multiple of symbol periods



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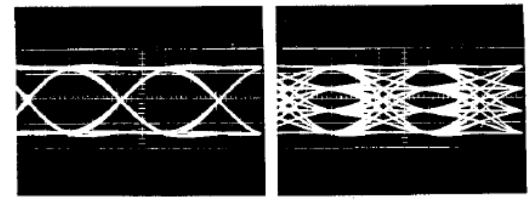
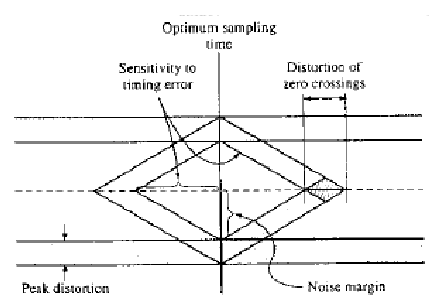
Postprocessing…

Eye Diagram Information: *Open and Clean Eye!*





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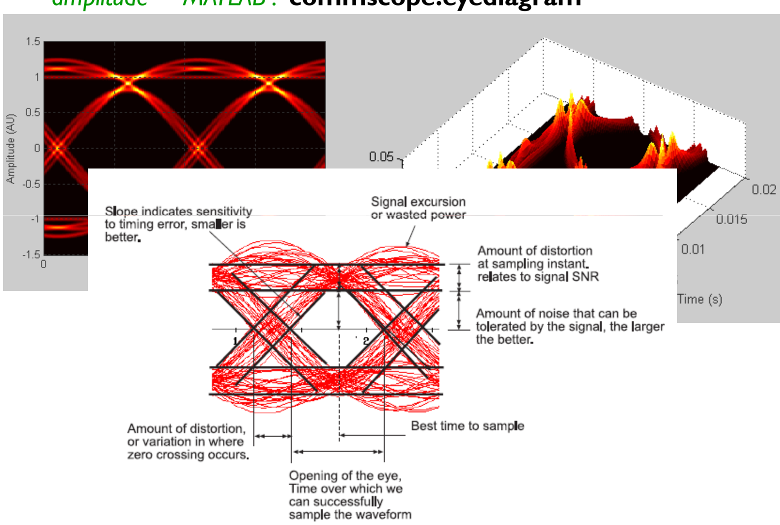
Postprocessing…

Eye Diagram Information: Vertical or Color → histograms to estimate pdf of amplitude *MATLAB :* **commscope.eyediagram**



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Postprocessing…

Constellation Diagram

π/4-DQPSK Example:

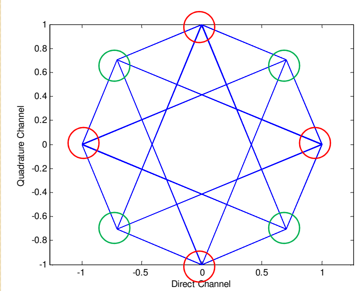
$$d(1), q(1), d(2), q(2), \ldots$$

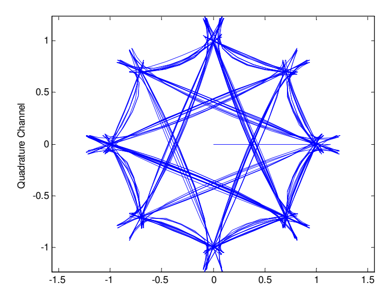
$$\theta(k) = \theta(k-1) + \varphi(k)$$

Information Symbols, $d(k)$ and $q(k)$	Differential Phase Shift, $\phi(k)$
1 1	$\pi/4$
0 1	$3\pi/4$
0 0	$-3\pi/4$
1 0	$-\pi/4$

After phase mapper d and q become d' and q' to make the proper phase

0010110111 → 0, -3π/4, -π, -3π/4, 0, π/4, …

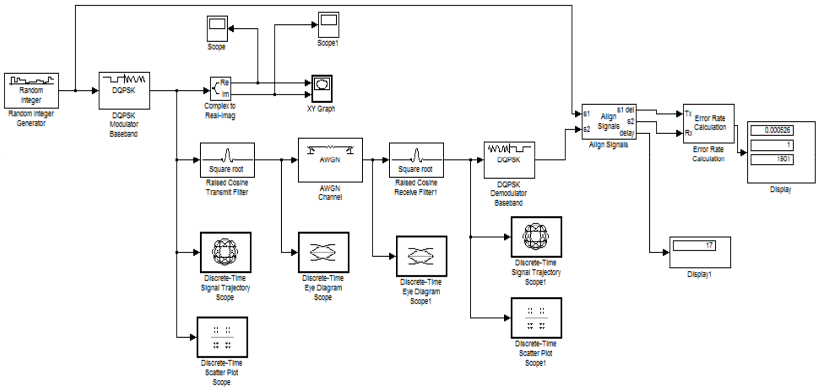




See commscope.ScatterPlot, …in Matlab

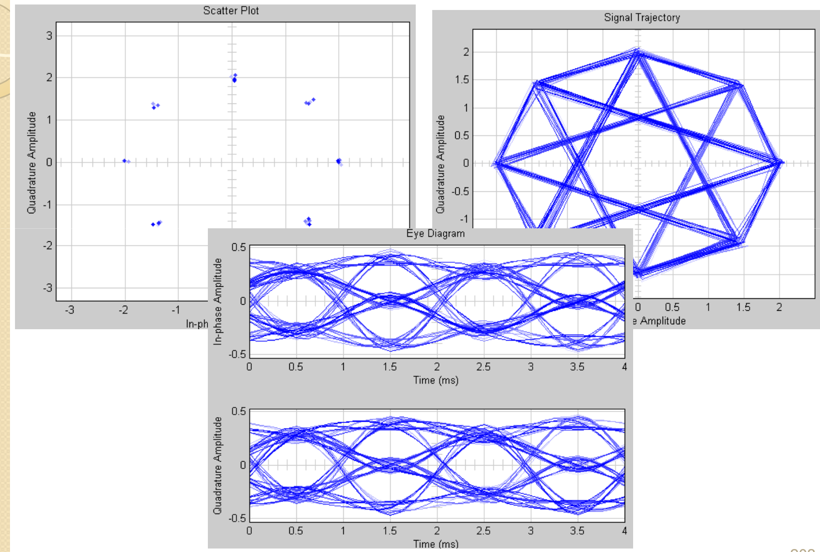
Postprocessing…

Simulink Example, Download it



Postprocessing…

Simulink Example 2



Postprocessing…

Estimation

Parametric: Based on a parametric model

Non-parametric: Direct manipulation of data

Good estimator:

Unbiased $E(\hat{A}-A)=0 \quad E(\hat{A})=A$

Consistent $E(\hat{A}-A)^2 \to 0$

pdf Estimation → Histogram, …

Histogram is a nonparametric pdf estimator

- *Assume n iid samples $X_1,\ldots X_n$ of an unknown pdf is given*
- *Divide the range into bins of width h and origin x_0:*

$$B_j=[x_0+(j-1)h, x_0+jh) \quad j\in \mathbf{Z}$$

- *Assume B_j count is n_j*
- *Normalize $f_j=n_j/nh$*
- *Draw bars of height f_j for bin B_j*

Postprocessing…

pdf Estimation → Histogram, …

If b_j is the center of bin B_j $\quad B_j=[b_j-h/2, b_j+h/2) \quad j\in \mathbf{Z}$

histogram assigns to each x in Bj the same estimate of $\hat{f}_h(b_j)$ *for f*

$$\hat{f}_h(x)=\frac{1}{nh}\sum_{i=1}^{n}\sum_{j}1(X_i\in B_j)1(x\in B_j)$$

random variable

Motivation:

$$P_j=P(X\in B_j)=\int_{B_j} f(x)dx \approx \hat{f}_h(b_j)h=\frac{N(X\in B_j)}{n} \Rightarrow \hat{f}_h(b_j)=\frac{n_j}{nh}$$

Histogram depends on the width h and the origin x_0

Assuming x_0=0 and x being in B_j=[(j -1)h, jh) :

$$\hat{f}_h(x)=\frac{1}{nh}\sum_{i=1}^{n}1(X_i\in B_j)$$

And for some c_j in B_j :

$$f(x)=f(b_j)+f'(b_j)(x-b_j)+f''(c_j)(x-b_j)^2/2$$

Postprocessing…

pdf Estimation → Histogram, …

Bias:

Taylor expansion of f around a point that estimate equals the true value

$$E[f(x)-\hat{f}_h(b_i')] = f'(b_i')(x-b_j') + f''(c_j)(x-b_j')^2/2$$

ignore the second term for small h

Bias is zero for some point in the bin

If h is small this point is approximately the center

Variance:

n_j number of samples in bin B_j out of n samples has Binomial distribution $B(n, P_j)$

Variance of n_j will be $np_j(1-p_j)$ → Variance of $f_j = np_j(1-p_j)/(nh)^2$

$$\sigma^2_{\hat{f}_h(x)} \approx \frac{f(x)h(1-f(x)h)}{nh^2} = \frac{f(x)(1-f(x)h)}{nh}$$

For small h: $\sigma^2_{\hat{f}_h(x)} \approx \dfrac{f(x)}{nh}$

decreasing h increases the variance while decreases the bias!

Postprocessing…

pdf Estimation → Histogram, …

Variance – Bias trade-off

$$MSE[\hat{f}_h(x)] = E[\hat{f}_h(x) - f(x)]^2 = \text{variance} + \text{Bias}^2$$

$$\approx f(x)/nh + [f'(x)]^2(x-b_j)^2$$

$$MISE[\hat{f}_h(x)] = E\left[\int_{-\infty}^{\infty}[\hat{f}_h(x)-f(x)]^2 dx\right] \approx \frac{1}{nh} + \frac{h^2}{12}\left\|f'^2\right\|_2$$

Optimal h:

$$\frac{\partial MISE}{\partial h} = 0 \Rightarrow h_{opt} = \left[\frac{6}{n\left\|f'^2\right\|_2}\right]^{1/3} \sim n^{-1/3}$$

But f is unknown and f' cannot be calculated

As a rule of thumb, assume f is normal:

$$\left\|f'^2\right\|_2 = 1/4\sqrt{\pi}$$

$$h_{opt} \approx 3.5n^{-1/3}$$

Postprocessing…

Estimation → Histogram, …

We proved what is mentioned in the textbook + a stronger trade off relation

$$b_i - \tfrac{h}{2} < x[n] \le b_i + \tfrac{h}{2}, \quad n_i \text{ count}$$

$$\text{Normalize to } nh \rightarrow \text{Area}(B_i) = \hat{f}_X(b_i)h = (n_i h/nh)h = hn_i/n$$

$$E(f_X(b_i) - \hat{f}_X(b_i)) \approx f_X''(b_i)h^2/24$$

$$Var(\hat{f}_X(b_i)) \approx \frac{1}{nh} f_X(b_i)$$

h → 0 to be unbiased → increases the variance!

h=1/sqrt(n) a good choice

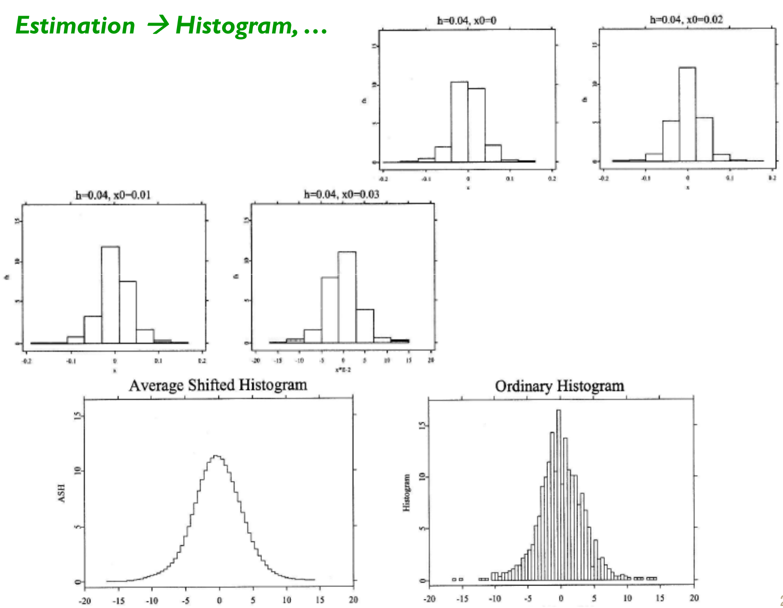
***ASH: Averaged Shifted Histogram :** solution to dependence on the origin*

Choose a proper h and generate M histograms with shifted bins:

$$B_j = [(j-1-\frac{k}{M})h, (j-\frac{k}{M})h) \quad j \in \mathbf{Z}, k = 0, 1, \ldots, M-1$$

Postprocessing…

Estimation → Histogram, …



Postprocessing…

Estimation → Power Spectral Density (PSD)

Classical non-Parametric method: Periodogram

Parametric methods: Model Based, AR, MA, ARMA,…

Trade-off between simple but with limited accuracy nonparametric and computationally demanding but more accurate parametric PSD

For ergodic processes if unlimited number of samples are available:

$$R_X(k) = \lim_{n\to\infty}\left[\frac{1}{2N+1}\sum_{n=-N}^{N} x(n)x(n+k)\right] \xleftrightarrow{F} S_X(f)$$

But in practice available samples are limited and noisy…

For instance using AR models helps to achieve better estimates when applicable

$$x(n) = \sum_{k=1}^{p} a_k x(n-k) + w(n) \rightarrow S_X(\omega) = \frac{\sigma_w^2}{\left|1-\sum_{k=1}^{p} a_k e^{-jk\omega}\right|^2}$$

Postprocessing…

Estimation → Power Spectral Density (PSD) …

For N data samples: $x(n), n=0,\ldots,N-1$, assuming x is zero outside [0,N)

$$\hat{R}_X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)x(n+k) = \frac{1}{N}\sum_{n=0}^{N-k-1} x(n)x(n+k)$$

The estimation will be biased because of the excluded samples (upper limit of the sum)

$$\hat{R}_X(k) = \frac{1}{N} x(k) * x(-k) \qquad \hat{S}_X(\omega) = \sum_{k=-N+1}^{N+1} \hat{R}_X(k) e^{-jk\omega}$$

$$\hat{S}_X(\omega) = \frac{1}{N} X(j\omega)X^*(j\omega) = \frac{1}{N}\left|X(j\omega)\right|^2 \longleftarrow \textbf{Periodogram}$$

Where $X(\omega) = \sum_{k=0}^{N-1} x(k)e^{-jk\omega}$ is the DTFT of x

Matlab

```
Sx=abs(fft(x(n1:n2))).^2/(n2-n1-1)
```

periodogram function

Postprocessing…

Estimation → Power Spectral Density (PSD) … Periodogram…

Using N-point FFT is very efficient, but not unbiased and not consistent!

$$\hat{S}(k\Delta f) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]\exp(-j2\pi k\Delta f n)\right|^2, \quad \Delta f = \frac{f_s}{N}, k = 0,\ldots, N-1$$

Bias is because of the finite length of data and approaches zero as N → ∞

$$E[\hat{S}_X(\omega)] = \sum_{k=-N+1}^{N+1} E[\hat{R}_X(k)]e^{-jk\omega}$$

$$= \sum_{k=-N+1}^{N+1} E[\frac{1}{N}\sum_{n=0}^{N-k-1} x(n)x(n+k)]e^{-jk\omega}$$

$$= \sum_{k=-N+1}^{N+1} \frac{N-|k|}{N} R_X(k)e^{-jk\omega}$$

$$= S_X(\omega) * W_B(\omega) \neq S_X(\omega)$$

Bartlett (triangular) window

$$W_B(\omega) = \frac{1}{N}\left[\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right]^2$$

Bias → 0 as N increases W_B = sinc^2 → delta function

$$E[\hat{S}_X(\omega) - S_X(\omega)] \xrightarrow[N\to\infty]{} 0$$

Asymptotically unbiased

Postprocessing…

Estimation → Power Spectral Density (PSD) … Periodogram…

For WGN of variance σ^2 it is unbiased!

$$E[\hat{S}_X(\omega)] = S_X(\omega) * W_B(\omega)$$

$$= \int_{-\pi}^{\pi} \sigma_X^2 \frac{1}{N}\left[\frac{\sin(\lambda)N/2)}{\sin(\lambda)/2)}\right]^2 d\lambda / 2\pi = \sigma_X^2$$

Periodogram resolution is also defined by the main lobe half power width of the rectangular window = 0.89 (2π/N)

Resolution is inversely proportional to N, bigger N gives better resolution

Tones closer than resolution cannot be resolved

Main Difficulty is the variance:

It is difficult to calculate the variance of S_X for general x

It depends on the 4th order moments of x:

Postprocessing…

Estimation → Power Spectral Density (PSD) … Periodogram…

$$E[\hat{S}_X(\omega_1)\hat{S}_X(\omega_2)] =$$

$$\frac{1}{N^2}\sum_m\sum_n\sum_p\sum_q E[x(m)x(n)x(p)x(q)]e^{-j[\omega_1(m-n)+\omega_2(p-q)]}$$

For WGN case, expectation is zero unless indices are equal in pairs, and :

$$E[x(m)x(n)x(p)x(q)] = \sigma_X^4$$

$$\mathrm{var}[\hat{S}_X(\omega)] = E[\hat{S}^2{}_X(\omega)] - \left[E[\hat{S}_X(\omega)]\right]^2 = \ldots = \sigma_X^4\left\{1+\left(\frac{\sin 2\pi f N}{N\sin 2\pi f}\right)^2\right\}$$

$$N\to\infty \Rightarrow \mathrm{var}[\hat{S}_X(\omega)] \to [S_X(\omega)]^2 \quad \textit{Independent of } N$$

For a non-white signal, imagine an LTI system to make the S_x from WGN case

$$\lim_{N\to\infty}\mathrm{var}[\hat{S}_X(\omega)] = [S_X(\omega)]^2$$ *Inconsistent estimator*

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Postprocessing…

Estimation → Power Spectral Density (PSD) … Periodogram…

If no windowing is applied to data samples, rectangular window is actually applied

Using windowed data:

$$\hat{S}(k\Delta f) = \frac{1}{U}\left|\sum_{n=0}^{N-1} x[n]w[n]\exp(-j2\pi k\Delta f n)\right|^2, \quad U = \sum_{n=0}^{N-1} w^2[n]$$

U is the energy in the data window

For rectangular U=N

A window more impulse-like (in f domain) might be used …

Smaller Side lobes vs. narrower main lobe 3dB width

Rectangular(0.89(2π/N), -13dB), Triangular(1.28 (2π/N), -27dB)

Hanning(1.30 (2π/N) , -32dB) , Hamming(1.44 (2π/N), -43dB)

Using other windows like Hanning, decreases the leakage but decreases the resolution too

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Postprocessing…

Estimation → Power Spectral Density (PSD) … Periodogram…

Application Related Recommendation for Window Type (National Instrument)

http://zone.ni.com/devzone/cda/tut/p/id/4844

Type of Signal	Window
Transients whose duration is shorter than the length of the window	Rectangular
Transients whose duration is longer than the length of the window	Exponential, Hanning
General-purpose applications	Hanning
Spectral analysis (frequency-response measurements)	Hanning (for random excitation), Rectangular (for pseudorandom excitation)
Separation of two tones with frequencies very close to each other but with widely differing amplitudes	Kaiser-Bessel
Separation of two tones with frequencies very close to each other but with almost equal amplitudes	Rectangular
Accurate single-tone amplitude measurements	Flat top
Sine wave or combination of sine waves	Hanning
Sine wave and amplitude accuracy is important	Flat top
Narrowband random signal (vibration data)	Hanning
Broadband random (white noise)	Uniform
Closely spaced sine waves	Uniform, Hamming
Excitation signals (hammer blow)	Force
Response signals	Exponential
Unknown content	Hanning

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Postprocessing…

Estimation → PSD …

Periodogram…

Variance reduction techniques

Split the data length N to K segments of length M, Compute FFT and Average

$$\hat{S}_X(k\Delta f) = \frac{1}{K}\sum_{i=1}^{K}\left\{\frac{1}{U}\left|\sum_{n=0}^{M-1} x[n]w[n]\exp(-j2\pi k\Delta f n)\right|^2\right\}, \quad \Delta f = \frac{f_s}{M}, k = 0,\ldots, M-1$$

For independent segment periodogram:

$$\mathrm{var}(\hat{S}_X(k\Delta f)) = \frac{1}{K}S_X^2(k\Delta f)$$

Resolution decreases → if large K is used → N is usually fixed

We need overlap but it kills the independence and variance will be more than …

Windowing helps to make the segments more uncorrelated → Hanning

Overlap length has also an optimal value

for Gaussian signals minimum variance is achieved 65% overlap (Hanning)

Postprocessing…

Estimation → Gain, Delay and SNR

x(t) reference signal, d(t) signal dependent internal distortion (for instance ISI)

n(t) external additive noise, y(t) is the measurement signal

$$y(t) = Ax(t-\tau) + n(t) + d(t)$$

Total noise power in MSE sense

$$\varepsilon(A,\tau) = E\left\{\left|y(t) - Ax(t-\tau)\right|^2\right\} = P_y + A^2 P_x - 2AR_{xy}(\tau)$$

$$\tau_{op} = \arg\max R_{xy}(\tau)$$

$$\frac{d\varepsilon(A,\tau)}{dA} = 0 \rightarrow A_{op} = R_{xy}(\tau_{op}) / P_x$$

$$N = P_y - R_{xy}^2(\tau_{op}) / P_x, \quad S = A_{op}^2 P_x$$

$$SNR = \frac{\rho^2}{1-\rho^2}, \quad \rho = \frac{R_{xy}(\tau_{op})}{\sqrt{P_x P_y}}$$

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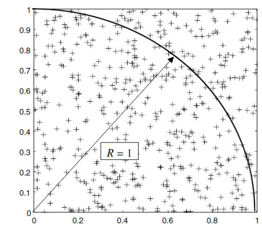
Postprocessing…

Estimation → BER

Monte Carlo Methods

Relative Frequency to estimate probability

$$P(A) = \lim_{N\rightarrow\infty} \frac{N_A}{N}$$



2500 trial
η={3.0960, 3.0720, 2.9920, 3.1600, 3.0480}

Application in communications: BER estimation

Case of AWGN: N_A has Binomial distribution, $P(N_A=N_e)$ can be calculated as:

$$P(N_e) = \binom{N}{N_e} P_e^{N_e} (1-P_e)^{N-N_e}$$

$$E(N_e) = NP_e \quad \sigma_{N_e}^2 = NP_e(1-P_e)$$

$$E(\widehat{P}_e) = E(N_e)/N = P_e$$

$$\sigma_{\widehat{P}_e}^2 = \sigma_{N_e}^2 / N^2 = \frac{P_e(1-P_e)}{N}$$

Unbiased and Consistent

EX6:
8.1,8.2,8.4,8.10,
8.12